

The Complexity of Tensor Rank

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Abstract We show that determining the rank of a tensor over a field has the same complexity as deciding the existential theory of that field. This implies earlier **NP**-hardness results by Håstad (J. Algorithm. **11**(4), 644–654 1990). The hardness proof also implies an algebraic universality result.

Keywords Computational complexity · Existential theory of fields · Tensor rank

1 Introduction

As computer scientists we can think of tensors as multi-dimensional arrays; 2-dimensional tensors correspond to (traditional) matrices, and a 3-dimensional tensor can be written as $T = (t_{i,j,k}) \in \mathbb{F}^{d_1 \times d_2 \times d_3}$. We will work over various fields, including \mathbb{Q} , \mathbb{R} , and \mathbb{C} , as well as GF_p . The rank of a matrix M (over some field \mathbb{F}) can be defined as the smallest k so that M is the sum of k matrices of rank 1, where a matrix of rank 1 is a matrix that can be written as $x \otimes y$, where x and y are one-dimensional vectors (over \mathbb{F}), and \otimes is the Kronecker (tensor, outer) product. The rank of a tensor can be defined similarly: a 3-dimensional tensor T has (*tensor*) rank at most k (over \mathbb{F}) if it is the sum of at most k rank-1 tensors, where a rank-1 tensor is a tensor of the form $x \otimes y \otimes z$ (over \mathbb{F}).

Håstad [12] showed that determining the tensor rank over \mathbb{Q} is an **NP**-hard problem; as Hillar and Lim [13] point out, his proof can be (mildly) adjusted to yield that the tensor rank problem remains **NP**-hard over \mathbb{R} and \mathbb{C} ; this is not immediate, since, tensor rank can vary depending on the underlying field (this is a well known fact; we will also see an example later on). This may suggest that tensor rank problems are equally intractable. Our goal in this paper is to show that this is not the case, and that the complexity of the tensor rank problem ranges wildly, as we consider different underlying fields.

For a field \mathbb{F} , let $\text{ETh}(\mathbb{F})$ be the set of true existential first-order statements over \mathbb{F} , sometimes known as the *existential theory of \mathbb{F}* . For example, letting $\varphi(c) := (\exists x)[x^2 = c]$, we have that $\varphi(2) \notin \text{ETh}(\mathbb{Q})$, but $\varphi(2) \in \text{ETh}(\mathbb{R})$, $\text{ETh}(\mathbb{C})$, and $\varphi(-1) \notin \text{ETh}(\mathbb{Q})$, $\text{ETh}(\mathbb{R})$, and $\varphi(-1) \in \text{ETh}(\mathbb{C})$. Our main result is that the tensor rank problem over \mathbb{F} is polynomial-time equivalent to the existential theory of \mathbb{F} .

Theorem 1.1 *Let \mathbb{F} be a field. Given a statement φ in $\text{ETh}(\mathbb{F})$, the existential theory of \mathbb{F} , we can in polynomial time construct a tensor T_φ and an integer k so that φ is true over \mathbb{F} if and only if T has tensor rank at most k over \mathbb{F} .*

The existential theory of any finite field is **NP**-complete, so Theorem 1.1 implies Håstad’s result that the tensor rank problem is **NP**-complete over finite fields [12]. If we use $\exists\mathbb{Q}$, $\exists\mathbb{R}$, and $\exists\mathbb{C}$ for the computational complexity class associated with deciding $\text{ETh}(\mathbb{Q})$, $\text{ETh}(\mathbb{R})$, and $\text{ETh}(\mathbb{C})$, respectively, then we can rephrase Theorem 1.1 as saying that the tensor rank problem is $\exists\mathbb{Q}$ -complete over the rationals, $\exists\mathbb{R}$ -complete over the reals, and $\exists\mathbb{C}$ -complete over the complex numbers.¹

While none of these complexity classes have been placed exactly with respect to traditional complexity classes, we do know that

$$\mathbf{NP} \subseteq \exists\mathbb{C} \subseteq \exists\mathbb{R} \subseteq \mathbf{PSPACE}.$$

The lower bound is folklore [8, Proposition 8].² The inclusion $\exists\mathbb{C} \subseteq \exists\mathbb{R}$ follows from the standard encoding of complex numbers as pairs of reals, and the upper bound of **PSPACE** on $\exists\mathbb{R}$ is due to Canny [9].

$\exists\mathbb{R}$ appears to contain problems harder than problems in **NP** or $\exists\mathbb{C}$: even a—seemingly simple—special problem in $\exists\mathbb{R}$ such as the sum of square roots problem has not been located in the polynomial-time hierarchy (see [1]). On the other hand, Koiran [16] showed that $\exists\mathbb{C} \subseteq \mathbf{AM}$, where **AM** is the class of Arthur-Merlin games, which is known to lie in $\Sigma_2^{\mathbf{P}}$, the second level of the polynomial-time hierarchy.³ This

¹The complexity class $\exists\mathbb{R}$ was introduced explicitly in [24, 26] and some other papers, but other researchers probably thought of $\text{ETh}(\mathbb{R})$ as a complexity class before, e.g., Shor [28], and Buss, Frandsen and Shallit [8].

²We are not aware of any stronger lower bounds on $\exists\mathbb{F}$ for *any* field \mathbb{F} . If we allow rings, then $\exists\mathbb{Z}$, for example, is undecidable, its complexity equivalent to the halting problem \emptyset' . This was shown in a famous series of results by Davis, Robinson, and Matiyasevic [10, 19].

³Koiran’s result assumes the generalized Riemann hypothesis (GRH); as far as we know there is no unconditional upper bound on $\exists\mathbb{C}$ other than **PSPACE**.

\mathbb{F}	complexity of tensor rank over \mathbb{F}	lower bound	upper bound
GF_p	NP -complete [12]		
\mathbb{C}	$\exists\mathbb{C}$ -complete	NP [12, 13]	AM $\subseteq \Sigma_2^P$ [16]
\mathbb{R}	$\exists\mathbb{R}$ -complete	NP [12, 13]	PSPACE [9]
\mathbb{Q}	$\exists\mathbb{Q}$ -complete	$\exists\mathbb{R}$ -hard	\emptyset'

Fig. 1 Complexity of the tensor rank problem over various rings. Previously all these problems were known to be **NP**-hard using Håstad’s argument [12, 13]

suggests that the tensor rank problem over \mathbb{C} may be significantly easier to solve (if still hard) than the tensor rank problem over \mathbb{R} .

The complexity of $\exists\mathbb{Q}$ is open, it is not even known (or expected) to be decidable. The currently best result in that direction is the undecidability of the $\exists\forall$ -theory of \mathbb{Q} , using definability results for \mathbb{Z} over \mathbb{Q} in the footsteps of Julia Robinson [15, 21]. Any decidability results for the tensor rank problem over \mathbb{Q} would, by our reduction, imply rather surprising decidability results for $\exists\mathbb{Q}$.⁴ We do know, however, that $\exists\mathbb{R} \subseteq \exists\mathbb{Q}$, since deciding the feasibility of a set of strict polynomial inequalities is hard for $\exists\mathbb{R}$ [26], and lies in $\exists\mathbb{Q}$.

Figure 1 summarizes our results for various fields. We note in particular that the upper bounds imply that there are (at least in principle) algorithms for solving the tensor rank problem over finite fields, \mathbb{R} and \mathbb{C} .

There are many computational problems related to tensors, and, as Hillar and Lim [13] showed compellingly, most of them are hard. Many of their hardness results are **NP**-hardness proofs via direct reductions from **NP**-complete problems, however, in one or two cases, they reduce from an $\exists\mathbb{R}$ -complete problem, and in those cases they also get $\exists\mathbb{R}$ -completeness results (even though they do not state this explicitly); in particular, testing whether 0 is an eigenvalue of a given tensor over \mathbb{R} is $\exists\mathbb{R}$ -complete (see Example 2.5 for a correction of their proof).

Our point is that it is important to capture the computational complexity of these algebraic problems more precisely than saying that they are **NP**-hard, since there may be a significant variance in their hardness (from $\exists\mathbb{C}$, close to **NP**, to $\exists\mathbb{R}$, probably closer to **PSPACE**, to $\exists\mathbb{Q}$, likely undecidable). For $\exists\mathbb{R}$, there already is a sizable number of complete problems, starting with Mnëv’s universality theorem showing that stretchability of pseudoline arrangements is complete for $\exists\mathbb{R}$ [20, 22, 28], but also including the rectilinear crossing number [4], segment intersection graphs [18] and many others. Less is known about $\exists\mathbb{Q}$, and $\exists\mathbb{C}$.

Our proof of Theorem 1.1 will work via a minimum rank problem for matrices with multilinear entries; versions of this problem were previously studied by Buss, Frandsen and Shallit [8]. We also show that both the minimum rank problem and the tensor rank problem exhibit algebraic universality. Algebraic universality implies that solutions to a problem may require algebraic numbers of high complexity.

⁴If \mathbb{Z} had an existential definition in \mathbb{Q} , then it would follow that $\exists\mathbb{Q} \equiv \exists\mathbb{Z} \equiv \emptyset'$. Koenigsmann [15] gives some evidence that there is no such definition (implying that his universal definition of \mathbb{Z} in \mathbb{Q} is optimal), however, there may be other routes towards the undecidability of $\exists\mathbb{Q}$, and it may be undecidable without being as hard as \emptyset' .

Remark 1.2 Shitov [27] has recently shown a stronger result—the complexity of the tensor rank over an *integral domain* is the same as the complexity of the existential theory of that integral domain.

2 Definitions and Tools

2.1 Tensors

A (3-dimensional, rational) *tensor* is an array $T = (t_{ijk})_{i,j,k=1}^{d_1,d_2,d_3} \in \mathbb{Q}^{d_1 \times d_2 \times d_3}$. Lower dimensional subarrays of a tensor are known as *fibres* (one dimension) and *slices* (two dimensions). We denote subarrays by using “:” instead of a variable, e.g., $t_{:jk}$ is a column-fibre of T , and $t_{::k}$ is a frontal slice. See [17] for a survey and additional notation.

We will use the symbol \otimes for the tensor (Kronecker, outer) product: for two vectors $u \otimes v$ is a matrix with entries $(u \otimes v)_{ij} = u(i)v(j)$, for three vectors $u \otimes v \otimes w$ is a tensor with entries $(u \otimes v \otimes w)_{ijk} = u(i)v(j)w(k)$. We say the tensor $u \otimes v \otimes w$ has *rank 1* unless it consists of zeros only, in which case it has rank 0. If a tensor T can be written as a sum of at most r rank-1 tensors, we say T has rank at most r . If $T = T_1 + \dots + T_r$, and each T_i has rank at most 1, we call $(T_i)_{i=1}^r$ a (*rank- r*) *expansion* of T .

The following two results are adapted from the conference version of Håstad’s paper [11]; in the journal version [12] they were replaced by references to other papers.

Lemma 2.1 (Håstad [11]) *Suppose $T = (t_{ijk})$ is a tensor of rank r (over some field), and the slice $M = (t_{::k_1})$ has rank 1, so $M = u_1 \otimes v_1$ for some u_1, v_1 . Then T can be written as $T = \sum_{\ell=1}^r u_\ell \otimes v_\ell \otimes w_\ell$.*

In other words, T has a rank- r expansion using the slice M as one of the rank-1 terms.

Lemma 2.2 (Håstad [11]) *Suppose $T = (t_{i,j,k})$ is a tensor of rank r (over some field), and there is a set of linearly independent slices $M_h = (t_{::h})$ of rank 1, so $M_h = u_h \otimes v_h$, for $h \in H$. Then T can be written as $T = \sum_{\ell=1}^r u_\ell \otimes v_\ell \otimes w_\ell$.*

In other words, if we have a set of linearly independent, rank-1 slices of a tensor, we can always assume that they occur in a minimum rank expansion of the tensor.

2.2 Logic and Complexity

Over a field (or ring) \mathbb{F} we can define the existential theory $\text{ETh}(\mathbb{F})$ of \mathbb{F} as the set of all true existential first-order sentences in \mathbb{F} . We work over the signature $(0, 1, +, *)$ and allow equality as predicate (for \mathbb{Q} and \mathbb{R} we can define order from that: $x \geq 0$ if and only if $(\exists y_0, y_1, y_2, y_3)[x = y_0^2 + y_1^2 + y_2^2 + y_3^2]$, using Lagrange’s theorem for \mathbb{Q}).

Lemma 2.3 (Buss et al. [8]) *Suppose \mathbb{F} is a field (a commutative ring without zero divisors is sufficient). Given a first-order existential sentence over \mathbb{F} one can construct (in polynomial time) a family of (multivariate) polynomials p_1, \dots, p_n with integer coefficients so that φ is true if and only if $(\exists x)[p_1(x) = 0 \wedge \dots \wedge p_n(x) = 0]$ is true over \mathbb{F} . If \mathbb{F} is not algebraically closed, then we can assume that $n = 1$.*

We write $\exists\mathbb{F}$ for the complexity class which is formed by taking the polynomial-time downward closure of $\text{ETh}(\mathbb{F})$. Lemma 2.3 then says that testing feasibility of a system of polynomial equations over \mathbb{F} is *complete* for the complexity class $\exists\mathbb{F}$, that is, it is *hard* for the complexity class (every problem in the class reduces to it), and it lies in the class (feasibility of a polynomial system over \mathbb{F} can be tested in $\exists\mathbb{F}$).⁵ We are particularly interested in $\mathbb{F} \in \{\text{GF}_p, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. We discussed relationships between these complexity classes and traditional complexity classes in the introduction.

Since a polynomial with integer coefficients can be calculated via a sequence of sums and products of variables and constants 1 and -1 , the following result follows immediately from Lemma 2.3.

Lemma 2.4 *Let \mathbb{F} be a field (or commutative ring without zero divisors). Deciding whether a system of equations of the types $x_i = x_j + x_k$, $x_i = x_j x_k$, $x_i = x_j$, and $x_i = 1$, is solvable over \mathbb{F} is complete for $\exists\mathbb{F}$.*

Call a such system of equations a *quadratic system*.

Let us illustrate $\exists\mathbb{R}$ -completeness with an example relevant to tensors. This corrects an example from Hillar and Lim [13, Remark 2.3].

Example 2.5 (Hillar and Lim [13]) A tensor $T = (t_{i,j,k})_{i,j,k=1}^{n,n,n}$ has *eigenvalue* λ if there is a non-zero vector x , the *eigenvector*, so that

$$\sum_{i,j=1}^{n,n} t_{i,j,k} x_i x_j = \lambda x_k.^6$$

So $\lambda = 0$ is an eigenvalue of T if there is a non-zero vector x satisfying $\sum_{i,j=1}^{n,n} t_{i,j,k} x_i x_j = 0$, which is a homogenous quadratic system of equations, and, obviously every homogenous quadratic system can be written in this form. So deciding whether a tensor has 0 as an eigenvalue is computationally equivalent to deciding whether a homogenous quadratic system has a non-trivial solution. This problem is sometimes called H_2N (for Hilbert's homogenous Nullstellensatz), and, over \mathbb{R} , was shown to be $\exists\mathbb{R}$ -complete in [25].⁷ Thus, deciding whether 0 is an eigenvalue of a tensor T over \mathbb{R} is $\exists\mathbb{R}$ -complete. Hillar and Lim [13, Remark 2.3] also sketch a

⁵In other models, e.g., the Blum-Shub-Smale model [7] this was well-known earlier.

⁶There are other definitions of eigenvalues for tensors as well.

⁷The proof in [25] yields a quartic systems, but that can be reduced to quadratic, by removing the final (unnecessary) squaring operation.

proof of the $\exists\mathbb{R}$ -completeness of $H_2\mathbb{N}$, but their proof of hardness of the quadratic homogenous system is not correct; in their notation, they require $z^2 = \sum_{i=1}^n x_i^2$, but this cannot be guaranteed. For example, they would take the quadratic system $(x+2)^2 = 0$ and homogenize it as $x^2 + 4xz + 4z^2 = 0$ and require $x^2 = z^2$. While the original system has a non-trivial solution, $x = -2$, it is easy to see that the homogenized system only has the trivial solution $x = z = 0$. The hardness proof seems to require a non-uniform construction as in [25].

2.3 Algebraic Universality

A solution to a system of algebraic equations may have high complexity, e.g., consider $x_0 = 1$, $x_1 = x_0 + x_0$, $x_2 = x_1x_1$, \dots , $x_n = x_{n-1}x_{n-1}$. This system of $n + 1$ equations defines a number x_n requiring a bit expansion of exponential length. Similarly, one can define a linear system whose solution is an algebraic number of high degree. $\exists\mathbb{R}$ -completeness reductions often preserve this property, so that $\exists\mathbb{R}$ -complete problems require solutions of high complexity. For example, Bienstock and Dean [4, 5] showed that any straight-line drawing of a graph with the smallest number of crossings may require vertex coordinates of double-exponential precision. This is a very weak type of algebraic universality. A stronger variant would, for example, show that for any algebraic number there is a graph which contains that algebraic number (after some normalization). A much stronger type of universality result goes back to Mnëv [20] who showed that any basic semialgebraic set is homotopy (even stably) equivalent to the realization space of a pseudoline arrangement. That is, for every basic semialgebraic set Mnëv defines a pseudoline arrangement so that the space of straight-line realizations of that pseudoline arrangement is essentially the same as the basic semialgebraic set up to some form of algebraic equivalence. We will show a weaker type of algebraic universality for the tensor rank problem. To do this properly, we need a definition of the realization space of a rank- r tensor. For a 3-dimensional tensor $T \in \mathbb{Q}^{d_1 \times d_2 \times d_3}$, and integer r define the rank- r realization space of T as

$$\mathcal{R}(T, r) := \left\{ (u_1, v_1, w_1, \dots, u_r, v_r, w_r) : T = \sum_{\ell=1}^r u_\ell \otimes v_\ell \otimes w_\ell \right\}.$$

Obviously, $\mathcal{R}(T, r) \subseteq \mathbb{R}^{(d_1+d_2+d_3)r}$ is an *algebraic set*; that is, it can be written as the set of common roots of a family of multivariate polynomials (with integer coefficients).

We would like to show that every algebraic set (with integer coefficients) over \mathbb{R} is essentially the same as some $\mathcal{R}(T, r)$ for some T and r , but it seems to have too many degrees of freedom, so instead we work with

$$\mathcal{R}(T, S, r) := \left\{ (w_1, \dots, w_r) : T = \sum_{\ell=1}^r S_\ell \otimes w_\ell \right\},$$

where S is a family of r rank-1 matrices.

We need to make precise the notion of being “essentially the same”, we will use the notion of stable equivalence introduced by Richter-Gebert to uniformize various universality constructions [22, 23]. Stable equivalence implies homotopy equivalence, and it maintains complexity of algebraic points [23]. Two sets are *rationally equivalent* if there is a rational homeomorphism between the two sets. A set X is a *stable projection* of Y if

$$Y = \{(y, y') : y \in X, \langle p_i(y), y' \rangle = c_i, 1 \leq i \leq n\},$$

where the p_i are multivariate polynomials with integer coefficients, and the c_i are constants. Two sets are *stably equivalent* if they are in the same equivalence class with respect to stable projections and rational transformations.

We will show that for every algebraic set (with integer coefficients), there are T , S , and r so that the algebraic set is stably equivalent to $\mathcal{R}(T, S, r)$, so this, restricted, tensor rank problem is universal for algebraic sets. By using $\mathcal{R}(T, S, r)$ instead of $\mathcal{R}(T, r)$ we side-step the fact that the two Håstad lemmas do not yield stable equivalence: forcing a particular $u_i \otimes v_i$ to equal a slice of T changes the number of algebraic components of the solution set, so it cannot maintain homotopy equivalence.

3 Hardness of Tensor Rank

In this section we will see that the tensor rank problem over a field \mathbb{F} is complete for $\exists\mathbb{F}$. In the Blum-Shub-Smale model, the same proof shows that the tensor rank problem over \mathbb{F} is $\text{NP}_{\mathbb{F}}$ -complete. We will not discuss the Blum-Shub-Smale model in detail, and refer the reader to [6].

3.1 A Minimum Rank Problem

For a matrix A with entries being multinomial expressions in $\mathbb{F}[x_1, \dots, x_n]$, the minrank of M is the smallest (matrix) rank of A over \mathbb{F} achievable by replacing variables x_i with values in \mathbb{F} and evaluating the resulting expressions.

Definition 3.1 Let $\text{minrank}_{\mathbb{F}}(A)$ be the minimum rank of A (as a matrix over \mathbb{F}) over all possible assignments of values in \mathbb{F} to variables in A .

Buss, Frandsen and Shallit [8] showed that the minrank problem over \mathbb{F} is complete for $\exists\mathbb{F}$, even if entries are restricted to be in $\mathbb{F} \cup \{x_1, \dots, x_n\}$. We will show that the minrank problem is $\exists\mathbb{F}$ -hard for matrices of a very specific form which lends itself to be turned into a tensor rank problem.⁸

Suppose we are given a quadratic system S with m equations e_1, \dots, e_m ; we construct a square $3m \times 3m$ matrix A with affine entries whose minrank will be connected

⁸There is also a notion of minrank for matrices with entries in $\{+, -\}$. Given such a matrix is there a real matrix of rank at most 3 with that sign pattern? This problem turns out to be $\exists\mathbb{R}$ -hard as well [2, 3], but does not seem to be related to our minrank problem.

to the feasibility of S (see Definition 3.1 and Lemma 3.2 below for a precise statement). To simplify the statements and the proofs we make the following assumptions on the quadratic system:

- A1 No variable occurs more than once in an equation.
- A2 Any two equations share at most one variable.
- A3 If $w = uv$ is an equation in S then v occurs exactly twice in S and the other occurrence of v is in an equation of the form $v = z$.

Assumptions **A1** and **A2** are not restrictive since we can always “copy” a variable v to a variable v' using equation $v' = v$ (and then use v' in place of v). Assumption **A3** is not restrictive since we can replace an equation $w = uv$ by a pair of equations $v' = v$, $w = uv'$, where v' is a new variable.

The following 3×3 matrices are the main building block in our construction

$$\det \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 1 & 1 & c \end{pmatrix} = c - (a + b), \quad (1)$$

$$\det \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & a \\ -1 & b & 0 \end{pmatrix} = c - ab. \quad (2)$$

To construct the matrix A we first place 3×3 blocks on the diagonal as follows: The ℓ -th diagonal 3×3 block is given by

- the matrix in (1) if e_ℓ is of the form $c = a + b$,
- the matrix in (2) if e_ℓ is of the form $c = ab$,
- the matrix in (1) with $b = 0$ if e_ℓ is of the form $c = a$,
- the matrix in (1) with $b = 0$, $a = K$ if e_ℓ is of the form $c = K$, where K is a rational constant.

Note that equation e_ℓ is satisfied if and only if the determinant of the block is zero. Let R_u be the increasing list of rows that contain variable u and let C_u be the increasing list of columns that contain variable u . From assumption **A1** it follows that a 3×3 block contains at most one occurrence of u . Thus $|R_u| = |C_u|$ and u occurs at positions $(R_u[i], C_u[i])$ for $i = 1 \dots |R_u|$. Also note that for distinct variables u, v we have that R_u and R_v are disjoint (since in the matrices in (1) and (2) the variables are in different rows). Now we add a few more entries into the matrix A . For every variable u , for every $1 \leq j \neq k \leq |R_u|$ we add an entry $u - u_j$, with new variable u_j , at position $(R_u[j], C_u[k])$ in A . This completes the construction of matrix A .

Observation 1 The construction satisfies the following:

1. u occurs exactly at positions $R_u \times C_u$ and it always occurs with coefficient 1,
2. the non-zero entries of A outside of the diagonal 3×3 blocks are at indices $\bigcup_u R_u \times C_u$,
3. u_j only occurs in the $R_u[j]$ -th row and it always occurs with coefficient -1 ,
4. leaving out every 3rd row and every 3rd column of A (that is, rows and columns whose index is divisible by 3) yields the $2m \times 2m$ identity matrix.

The third item in Observation 1 follows from assumption **A3** and the form of the matrices in (1) and (2). Note that the only occurrence of a variable in a column whose index is not divisible by 3 must come from “ b ” in (2), that is, an equation of the form $c = ab$. The other occurrence of b is in a row whose index is divisible by 3 (using assumption **A3**). Since both occurrences of b are in rows whose index is divisible by 3 we have that $R_b \times C_b$ is in the left-out part of A . We showed that for every u either all entries of C_u or all entries of R_u are divisible by 3 and hence if we leave out every third column and every third row there will be no off-diagonal entries.

We have the following connection between the quadratic system S and its matrix A .

Lemma 3.2 *Assume that a quadratic system S satisfies assumptions **A1**, **A2**, and **A3**. Let A be the matrix corresponding to S . System S is solvable over \mathbb{F} if and only if $\text{minrank}_{\mathbb{F}}(A) = 2m$.*

Example 3.3 Before proving Lemma 3.2 let us illustrate the construction with an example. Let $S = \{u = xy, y = x, u = 2\}$. Then the matrix A corresponding to S is

$$\begin{pmatrix} 1 & 0 & u & 0 & 0 & 0 & 0 & 0 & u - u_1 \\ 0 & 1 & x & 0 & 0 & x - x_1 & 0 & 0 & 0 \\ -1 & y & 0 & 0 & 0 & y - y_1 & 0 & 0 & 0 \\ 0 & 0 & x - x_2 & 1 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & y - y_2 & 0 & 1 & 1 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & u - u_2 & 0 & 0 & 0 & 0 & 1 & u \end{pmatrix} \quad (3)$$

The quadratic system S encodes the equation $x^2 = 2$. This equation has a solution over \mathbb{R} and hence, by Lemma 3.2, $\text{minrank}_{\mathbb{R}}(A) = 6$. On the other hand the equation does not have a solution over \mathbb{Q} and hence, by Lemma 3.2, $\text{minrank}_{\mathbb{Q}}(A) \geq 7$.

Proof of Lemma 3.2 From Observation 1 (part 4) we have $\text{minrank}_{\mathbb{F}}(A) \geq 2m$.

Suppose that S has a solution σ with values in \mathbb{F} . For each variable u assign value $\sigma(u)$ to u and all u_i 's in A . Note that this assignment makes all entries outside the diagonal 3×3 blocks zero (since those entries are of the form $u - u_i$). Also note that each 3×3 block has rank 2 (since it contains a 2×2 identity matrix and has determinant equal to zero—here we use the fact that σ is a solution of S). The rank of a block diagonal matrix is the sum of the ranks of the blocks and hence $\text{minrank}_{\mathbb{F}}(A) = 2m$.

It remains to show that $\text{minrank}_{\mathbb{F}}(A) = 2m$ implies that S has a solution in \mathbb{F} . Let σ be an assignment with values in \mathbb{F} such that the rank of $\sigma(A)$ is $2m$. Consider the ℓ -th 3×3 diagonal block \hat{B} . Let \hat{A} be the matrix obtained from $\sigma(A)$ by leaving out every third row and every third column except for the column and the row with index 3ℓ . Note that \hat{A} is a $(2m + 1) \times (2m + 1)$ matrix and, by Observation 1 (part 4), if we

leave out the row and column with index $2\ell + 1$ from \hat{A} we get the identity matrix. Hence we have

$$\begin{aligned}\det(\hat{A}) &= \hat{A}_{2\ell+1,2\ell+1} - \sum_{i \neq 2\ell+1} \hat{A}_{i,2\ell+1} \hat{A}_{2\ell+1,i} \\ &= \det(\hat{B}) - \sum_{i \notin \{2\ell-1, 2\ell, 2\ell+1\}} \hat{A}_{i,2\ell+1} \hat{A}_{2\ell+1,i}.\end{aligned}$$

We have

$$\sum_{i \notin \{2\ell-1, 2\ell, 2\ell+1\}} \hat{A}_{i,2\ell+1} \hat{A}_{2\ell+1,i} = \sum_{i \neq \ell} A_{3i-2,3\ell} A_{3\ell,3i-2} + \sum_{i \neq \ell} A_{3i-1,3\ell} A_{3\ell,3i-1}. \quad (4)$$

Note that $A_{3\ell,3i-2} = 0$ for all $i \neq \ell$ since the first column in (1) and (2) does not contain any variables (also see Observation 1 (part 2)). If $A_{3\ell,3i-1} \neq 0$ then the i -th block contains a variable in the 2-nd column (and hence in the 3-rd row) and that variable also occurs in the 3-rd row of the ℓ -th block. If $A_{3i-1,3\ell} \neq 0$ then the i -th block contains a variable in the 2-nd row and that variable also occurs in the 3-rd column of the ℓ -th block. Thus if both $A_{3\ell,3i-1} \neq 0$ and $A_{3i-1,3\ell} \neq 0$ then e_i and e_ℓ would share two variables (occurring in the 2-nd and 3-rd row of the i -th block). This is impossible (because of assumption **A2**) and hence (4) has value 0. We conclude that $\det(\hat{A}) = \det(\hat{B})$.

Now \hat{A} has rank at most $2m$, since $\sigma(A)$ has rank $2m$, but dimension $(2m + 1) \times (2m + 1)$, so its columns are linearly dependent, and we conclude that

$$0 = \det(\hat{A}) = \det(\hat{B})$$

and hence the ℓ -th equation is satisfied by the assignment σ , for all $\ell \in [m]$. Thus σ is a solution of S in \mathbb{F} . \square

3.2 A Tensor Rank Problem

We are left with translating the minrank problem from the previous section into a tensor rank problem. Recall that given a quadratic system S we constructed a matrix A consisting of diagonal blocks (with constants and variable terms) and additional, affine entries in rows and columns divisible by 3.

Define a tensor T_A from A as follows:

- for every variable x in A let the partial derivative $A_x := \partial A / \partial x$ be a (frontal) slice of T ; $\partial A / \partial x$ is the matrix containing the coefficients of x in A ,
- add one final (frontal) slice A_1 containing all the constant values of A .

Note that if σ assigns a value in \mathbb{F} to each variable in A , then $\sigma(A) = A_1 + \sum_x \sigma(x) A_x$. Let n be the number of variables in A ; T_A is a $3m \times 3m \times n + 1$ tensor.

Lemma 3.4 *A has minrank at most $2m$ if and only if T_A has tensor rank at most $2m + n$.*

Proof If A has minrank $2m$, then there is a σ assigning $\sigma(x) \in \mathbb{F}$ to each variable x occurring in A so that the rank of $\sigma(A)$ is $2m$. Now $\sigma(A) = A_1 + \sum_x \sigma(x)A_x$, where the sum is over all n variables x occurring in A . In other words, $A_1 = \sigma(A) + \sum_x (-A_x)$. Since $\sigma(A)$ has matrix rank $2m$, it can be written as the sum of $2m$ rank-1 matrices, so A_1 can be written as the sum of $2m + n$ rank-1 matrices—each A_x has rank 1. Hence, every slice of T_A can be written using the A_x and the $2m$ rank-1 matrices summing up to A_1 , implying that T has tensor rank at most $2m + n$.

For the other direction, assume that T_A has tensor rank at most $2m + n$. We first observe that the n matrices A_x are linearly independent: Suppose that $\sum_x \lambda(x)A_x = 0$ for some vector λ . The matrix A contains two types of variables: the original variables u (from the quadratic system), and the additional variables u_j . Now any non-zero entry in A_u is unique in the sense that no other A_x has an entry in the same position, so $\lambda(u) = 0$ for the original variables. But then any non-zero entry in A_{u_j} is unique among the remaining matrices (belonging to the non-original variables), so $\lambda(u_j) = 0$ for all remaining variables, establishing $\lambda = 0$. Therefore, the A_x are linearly independent.

Lemma 2.2 now implies that T can be written using the A_x and $2m$ additional rank-1 tensors. So $T = \sum_x A_x \otimes z_x + \sum_{i=1}^{2m} u_i \otimes v_i \otimes w_i$, and, in particular,

$$A_1 = \sum_x \tau(x)A_x + \sum_{i=1}^{2m} B'_i,$$

where $\tau(x) = z_x(1)$, and $B_i = w_i(1)(u_i \otimes v_i)$, where the B_i are rank-1 matrices. In other words, $A_1 - \sum_x \tau(x)A_x = \sum_{i=1}^{2m} B_i$ has matrix rank at most $2m$. Setting $\sigma(x) := -\tau(x)$ we have that $A_1 + \sum_x \sigma(x)A_x$ has rank $2m$, and, moreover, equals $\sigma(A)$. But this shows that the minrank of A is at most $2m$, which is what we had to prove. \square

The following is a well-known result. For more results on tensor rank over various rings, see Howell [14].

Corollary 3.5 *There is a tensor T with $\text{rank}_{\mathbb{Q}}(T) > \text{rank}_{\mathbb{R}}(T)$.*

Proof Let A be the matrix from Example 3.5, and consider the tensor T_A constructed in Lemma 3.4. Then $\text{rank}_{\mathbb{Q}}(T_A) \geq 7 + 9 = 16$, while $\text{rank}_{\mathbb{R}}(T_A) = 6 + 9 = 15$. \square

We can now complete the proof of our main result.

Proof of Theorem 1.1 Lemmas 2.3 and 2.4 allow us to translate φ into a quadratic system S so that φ is true over \mathbb{F} if and only if S has a solution over \mathbb{F} . Lemma 3.2 translates S into a minrank problem over a matrix A , and Lemma 3.4 turns that into a tensor rank problem over \mathbb{F} . \square

3.3 Universality

Reviewing the hardness proofs carefully shows that they also yield algebraic universality. Let us start with the minrank problem:

Corollary 3.6 *For every algebraic set V specified using integer coefficients, we can find a matrix A whose entries are multilinear expressions in $\mathbb{F}[x_1, \dots, x_m]$, and an integer k so that V is stably equivalent to $\{(x_1, \dots, x_d) : \text{minrank}_{\mathbb{F}}(A) = k\}$.*

Proof Suppose we are given an algebraic set $V = \{(x_1, \dots, x_d) \in \mathbb{F}^d : p_1(x_1, \dots, x_d) = \dots = p_n(x_1, \dots, x_d) = 0\}$. We transform the system $p_1(x_1, \dots, x_d) = \dots = p_n(x_1, \dots, x_d) = 0$ into a quadratic system S (as in Lemma 2.4). While S may require additional variables, each of these is equal to a polynomial transformation of the x_i so that the realization space of S is stably equivalent to the original algebraic set V (in this case via a rational transformation). In the next step, we turn S into a matrix A with multilinear expressions over x_1, \dots, x_m , and an integer k as in Lemma 3.2 so that S is solvable if and only if $\text{minrank}_{\mathbb{F}}(A) = k$. Moreover, the variables of S are variables of A , though A may contain additional variables. However, those, as before, equal existing variables when $\text{minrank}_{\mathbb{F}}(A) = k$, so S is stably equivalent to $\{(x_1, \dots, x_d) : \text{minrank}_{\mathbb{F}}(A) = k\}$, and then, by transitivity, so is V . \square

In other words, the minrank problem for matrices with multilinear expressions over a field is universal for algebraic sets over that field. This gives us universality of the tensor problem as well.

Corollary 3.7 *For every algebraic set V we can find a tensor T , an integer r , and a family of r rank-1 matrices S so that V is stably equivalent to the realization space $\mathcal{R}(T, S, r)$.*

Proof By Corollary 3.6, the algebraic set V is stably equivalent to a minrank problem $\text{minrank}_{\mathbb{F}}(A) = k_A$ for matrix A and k_A as constructed in the proof of Lemma 3.2. From A we construct a $3m \times 3m \times (n+1)$ tensor T and an integer $k = 2m + n$, as in Lemma 3.4, so that $V \neq \emptyset$ if and only if the tensor rank of T is at most k . We know what the potential basis for T looks like: it consists of the n matrices A_{x_i} , the coefficient matrix of x_i , and $2m$ matrices B_i , two for each of the m blocks in the minrank problem (keeping first and second column in each block). Letting $S_i = A_{x_i}$ for $1 \leq i \leq n$, and $S_i = B_{i-n}$ for $n+1 \leq i \leq 2m+n$, we know that if T has tensor rank at most $k = 2m + n$, then it can be written using the S_i . Consider an element of the realization space

$$\mathcal{R}(T, S, r) = \left\{ (w_1, \dots, w_r) : T = \sum_{i=1}^r S_i \otimes w_i \right\}.$$

Recall that $t_{:,n+1} = A_1$, the matrix of constants from the minrank problem, so, as in the proof of Lemma 3.4 we know that

$$A_1 = \sum_{i=1}^r w_i [n+1] S_i,$$

which means $A_1 = \sum_{i=1}^n w_i [n+1] A_{x_i} + \sum_{i=n+1}^r w_i S_i$. This implies, as we argued in the lemma, that $-w_i [n+1]$ is the value of x_i in a solution to the minrank

problem. So $\mathcal{R}(T, S, r)$ is stably equivalent to the minrank problem, and, thus, to V (the conditions on the remaining w_i are affine, so they can be removed by a stable projection). \square

4 Open Questions

There are several natural follow-up questions suggested by the results of this paper. For example, what is the complexity of tensor rank for symmetric tensors? Is tensor-rank hard for a fixed rank (2 or 3 even) or is it fixed-parameter tractable? Over the complex numbers, Koiran's result places the problem at the second level of the polynomial hierarchy assuming the Generalized Riemann hypothesis is true. With the recent successes of exact algorithms for NP-hard problems, is there a way to make Koiran's result algorithmic? Is there a way to remove the assumption?

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