

CB's Answers, Asst. 2

CSC 172

8 Sept. 1997

Read the handout on "Mathematical Preliminaries".

1. How many different 5-card draw poker hands may be dealt from a 52 card deck?

52 choose 5, or $\binom{52}{5}$, or $52!/(47!5!)$.

2. How many different 5-card draw poker hands may be dealt from a 54 card deck, one with two identical jokers?

Since the number of permutations of the cards obeys the law in the handout for sets with identical members, it seemed to me that it would be $54!/(49!5!2!)$, but I much prefer the argument that there are $\binom{52}{5}$ ways to choose a hand without jokers, $\binom{52}{4}$ ways to choose a hand with 1 joker, and $\binom{52}{3}$ ways to choose a hand with 2 jokers. These two answers don't agree, so something's wrong with the first one. To be continued...

3. If you have an unfair coin with probability p of coming up heads, with p not necessarily equal to $1/2$, and thus $q = 1 - p$ probability of coming up tails, what is the probability of tossing the coin n times and getting k heads?

This is pretty easy to figure out from first principles, and in fact is also the general term of the *binomial probability density function*, an important one in statistics....

$$P(n, k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

4. Explain the fact that $11^4 = 14641$ in terms of Pascal's triangle.

Think of this as $(10 + 1)^4$, which because there are no carries can be written as $1(10000) + 4(1000) + \dots + 1$, which is just the binomial coefs multiplying the powers of 10. Because of pesky carries the phenomenon is obscured for 11^5 .

5. What is the relation between $a^{\log_c n}$ and $n^{\log_c a}$?

They're equal, just take log to base c of both sides.

6. Prove that F_{n+1} and F_n are relatively prime. This is easy once you prove (by induction) that with F_n a Fibonacci number, $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.

You can verify this relation for the first few F 's. Then you want to prove that thencefore, adjacent versions of the difference have the same magnitude and different signs, preserving the relationship. So we want

$$F_{n-1}F_n - 1 - F_nF_{n-2} = F_{n+1}F_{n-1} - F_nF_n$$

Now the trick is to expand each side in such a way as to have no terms below F_{n-2} .

$$F_{n-1}F_n - 1 - F_{n-1}F_{n-2} - F_{n-2}F_{n-2} = F_nF_{n-1} + F_{n-1}F_{n-1} - F_{n-1}F_{n-1} - 2F_{n-1}F_{n-2} - F_{n-2}F_{n-2},$$

so

$$-F_{n-1}F_{n-2} = F_nF_{n-1} - F_{n-1}F_{n-1} - 2F_{n-1}F_{n-2}$$

and

$$F_{n-1}F_{n-2} = F_nF_{n-1} - F_{n-1}F_{n-1},$$

$$F_{n-1}F_{n-2} + F_{n-1}F_{n-1} = F_nF_{n-1},$$

$$F_nF_{n-1} = F_nF_{n-1}.$$

(I don't claim this is the most elegant way to do this!)

Once the relation is proved, it is easy to see that F_n and F_{n+1} are relatively prime, since any common divisor would have to be a divisor of $(-1)^n$.

7. Express $\log_{10} x$ in terms of $\log_2 x$ and $\log_2 10$.

We did this already:

$$\log_{10} x = \frac{\log_2 x}{\log_2 10}.$$

8. If n is a 14-digit integer, will the value of n fit in a computer word with a capacity of 47 bits plus sign?

Yes indeed, since $\log_2 n < (\log_{10} n)/(\log_{10} 2) < 14/0.30103 < 47$

9. A prime number is an integer greater than one which has no exact divisors other than 1 and itself. Using this definition and mathematical induction, prove that every positive integer greater than one may be written as a product of prime numbers. (Hint: don't "induce" by adding 1 to n . Rather start your proof out: "If n is prime it is definitely the product of itself and 1. If it isn't prime...".

If n is prime it is definitely the product of itself and 1. If it isn't prime, it has two factors p and q such that $n = pq$ and both p and q are less than n . Use the inductive argument on p and q , noting that since they are decreasing with every application, there will come a time when they are either prime or 1.