

Recall: Truth in a model  $\mathcal{M} = (\mathcal{D}, I)$   
for a predication  $\pi(\alpha_1, \dots, \alpha_n)$ ,  
where  $\alpha_1, \dots, \alpha_n$  are constants:

$$\models_{\mathcal{M}} \pi(\alpha_1, \dots, \alpha_n) \text{ iff } \langle \alpha_1^I, \dots, \alpha_n^I \rangle \in \pi^I$$

### Interpreting functions & functional terms

e.g.,  $\text{age}^I = \{ \langle r, 16 \rangle, \langle j, 14 \rangle, \langle 14, 14 \rangle, \langle 16, 16 \rangle \}$

In general, for  $\alpha$  an  $n$ -place fcn. constant,  
let  $\alpha^I: \mathcal{D}^n \rightarrow \mathcal{D}$ , a total fcn,  
i.e., a single-valued relation on  $\mathcal{D}^{n+1}$   
(&  $\text{domain}(\alpha^I) = \mathcal{D}^n$ )

To interpret functional terms  $\alpha(\beta_1, \dots, \beta_n)$ ,  
extend  $I$  to  $T_I$ , "term assignment for"

$$T_I(\beta) = I(\beta) = \beta^I \text{ for } \beta \text{ a constant}$$

$$T_I(\alpha(\beta_1, \dots, \beta_n)) = \alpha^I(T_I(\beta_1), \dots, T_I(\beta_n))$$

e.g.,  $T_I(\text{age}(\text{Juliet})) = \text{age}^I(T_I(\text{Juliet}))$   
 $= \text{age}^I(\text{Juliet}^I)$   
 $= \{ \langle r, 16 \rangle, \langle j, 14 \rangle, \langle 14, 14 \rangle, \langle 16, 16 \rangle \} (j)$   
 $= 14$

e.g.,  $T_I(\text{age}(\text{age}(\text{Juliet})))$   
 $= \text{age}^I(T_I(\text{age}(\text{Juliet})))$   
 $= \text{age}^I(\text{age}^I(T_I(\text{Juliet})))$   
 $= \text{age}^I(\text{age}^I(\text{Juliet}^I))$   
 $= \text{age}^I(\text{age}^I(j)) = \text{age}^I(14) = 14$

Truth conditions for equality,  
given  $\mathcal{M} = (\mathcal{D}, I)$ :

$$\models_{\mathcal{M}} (\alpha = \beta) \text{ iff } T_I(\alpha) \overset{\text{meta-equality}}{=} T_I(\beta)$$

e.g.,  $\models_{\mathcal{M}} (\text{Juliet} = \text{Juliet})$   
 iff  $T_I(\text{Juliet}) = T_I(\text{Juliet})$   
 iff  $\text{Juliet}^I = \text{Juliet}^I \dots \text{TRUE, because}$   
 $I$  is a function!

### Truth conditions for predication

$\Pi(\alpha_1, \dots, \alpha_n)$  where  $\alpha_i$  may be functional terms:

$$\models_{\mathcal{M}} \Pi(\alpha_1, \dots, \alpha_n) \text{ iff } \langle T_I(\alpha_1), \dots, T_I(\alpha_n) \rangle \in \Pi^I$$

e.g.  $\models_{\mathcal{M}} \text{Loves}(\text{Romeo}, \text{age}(\text{Juliet})) \text{ ??}$

$$\text{iff } \langle T_I(\text{Romeo}), T_I(\text{age}(\text{Juliet})) \rangle \in \text{Loves}^I$$

$$\text{iff } \langle I(\text{Romeo}), \text{age}^I(T_I(\text{Juliet})) \rangle \in \text{Loves}^I$$

$$\text{iff } \langle r, \text{age}^I(j) \rangle \in \{ \langle r, j \rangle, \langle j, r \rangle, \langle s, b \rangle, \dots \}$$

$$\text{iff } \langle r, 14 \rangle \in \{ \langle r, j \rangle, \dots \} \quad \text{but not } \langle r, 14 \rangle$$

FALSE

### Truth conditions for sentences with $\neg, \wedge, \vee, \Rightarrow$ :

$$\models_{\mathcal{M}} \neg \varphi \text{ iff not } \models_{\mathcal{M}} \varphi$$

$$\models_{\mathcal{M}} (\varphi \wedge \psi) \text{ iff } \models_{\mathcal{M}} \varphi \text{ and } \models_{\mathcal{M}} \psi$$

...  $\vee$  ...      ... or ...

$$\models_{\mathcal{M}} (\varphi \Rightarrow \psi) \text{ iff not } \models_{\mathcal{M}} \varphi \text{ or } \models_{\mathcal{M}} \psi$$

Consider again the FO language containing constants Snoopy, Romeo, Juliet, 14, 16; Dog, Has-Tail; Loves; age

and the model  $(\mathcal{D}, I)$ ,  $\mathcal{D} = \{s, r, j, 14, 16, \dots\}$

$\text{Snoopy}^I = s$ ,  $\text{Romeo}^I = r$ , etc

$\text{Dog}^I = \{s, f, \dots\}$  but not  $r, j, 14, 16!$

$\text{Loves}^I = \{ \langle r, j \rangle, \langle j, r \rangle, \langle s, b \rangle, \dots \}$

e.g.,  $\models_{\mathcal{M}} \neg \text{Dog}(\text{Romeo}) \text{ ??}$

iff not  $\models_{\mathcal{M}} \text{Dog}(\text{Romeo})$  (also written  $\not\models_{\mathcal{M}}$ ) by the truth cond's of  $\neg$   
iff not  $\text{Romeo}^I \in \text{Dog}^I$   
iff not  $r \in \{s, f, \dots\}$  TRUE

e.g.,  $\models_{\mathcal{M}} (\neg \text{Dog}(\text{Romeo})) \Rightarrow \text{age}(\text{Juliet}) = 16 \text{ ??}$

iff not  $\underbrace{\models_{\mathcal{M}} \neg \text{Dog}(\text{Romeo})}_{\text{TRUE (see above)}} \text{ or } \models_{\mathcal{M}} \text{age}(\text{Juliet}) = 16$   
FALSE

iff FALSE or  $\underbrace{T_I(\text{age}(\text{Juliet}))}_{14 \text{ (seen earlier)}} = \underbrace{T_I(16)}_{16^I = 16}$   
FALSE

$\therefore$  FALSE

FALSE

## ALLOWING FOR VARIABLES

e.g.,  $P(A, x)$  ... "truth" in  $\mathcal{M} = (\mathcal{D}, \mathcal{I})$   
doesn't quite make sense

Introduce variable assignment function  $U$

$$U : \{\text{variables of our FO lang.}\} \rightarrow \mathcal{D}$$

e.g.,  $U(x) = r,$   
 $U(y) = 16,$  etc

or  $x^U = r,$   
 $y^U = 16, \dots$

Extend  $T_I$  to  $T_{IU}$ : (term assign. fcn)

$$T_{IU}(\alpha) = \alpha^I \text{ if } \alpha \text{ is an ind. constant}$$

$$= \alpha^U \text{ ————— " ————— variable}$$

$$T_{IU}(\alpha(\beta_1, \dots, \beta_n)) = \alpha^I(T_{IU}(\beta_1), \dots, T_{IU}(\beta_n))$$

e.g.,  $T_{IU}(\text{Romeo}) = \text{Romeo}^I = r.$

$$T_{IU}(x) = x^U = r$$

$$T_{IU}(\text{age}(\text{Romeo})) = \text{age}^I(T_{IU}(\text{Romeo}))$$

$$= \text{age}^I(r) = 16$$

$$T_{IU}(\text{age}(y)) = \text{age}^I(T_{IU}(y)) = \text{age}^I(y^U)$$

$$= \text{age}^I(16) = 16$$

For formulas involving variables,  
we speak of satisfaction conditions  
rather than truth conditions (but  
we'll come back to truth soon!)

Satisfaction conditions for equality  
and predication

$$\models_{\mathcal{M}} (\alpha = \beta) [U] \quad \text{"v.a. } U \text{ satisfies } (\alpha = \beta) \text{ relative to model } \mathcal{M} \text{"}$$

$$\text{iff } T_{IU}(\alpha) = T_{IU}(\beta)$$

$$\models_{\mathcal{M}} \pi(\alpha_1, \dots, \alpha_n) [U] \quad \text{"v.a. } U \text{ satisfies } \pi(\alpha_1, \dots, \alpha_n) \text{ relative to model } \mathcal{M} \text{"}$$

$$\text{iff } \langle T_{IU}(\alpha_1), \dots, T_{IU}(\alpha_n) \rangle \in \pi^I$$

e.g., for the Snoopy + Romeo & Juliet model  $\mathcal{M}$ ,  
and  $U(x) = r, U(y) = 16$

$$\models_{\mathcal{M}} \text{Loves}(\text{Romeo}, x) [U]$$

$$\text{iff } \langle T_{IU}(\text{Romeo}), T_{IU}(x) \rangle \in \text{Loves}^I$$

$$\text{iff } \langle r, r \rangle \in \text{Loves}^I, \text{ which is FALSE}$$

Satisfaction conditions for  $\neg, \vee, \wedge, \Rightarrow$ :

$\models_{\mathcal{M}} \neg \psi [U]$  iff not  $\models_{\mathcal{M}} \psi [U]$

$\models_{\mathcal{M}} (\psi \wedge \psi) [U]$  iff  $\models_{\mathcal{M}} \psi [U]$  and  $\models_{\mathcal{M}} \psi [U]$   
etc

### Satisfaction conditions for $\forall$

$\models_{\mathcal{M}} (\forall x \psi) [U]$  iff for all v.a.'s  $U'$  identical with  $U$  except possibly that  $U'(x) \neq U(x)$ ,  $\models_{\mathcal{M}} \psi [U']$  (scope)

Writing  $U_{x:d}$  for the assignment function that is the same as  $U$  except that  $U_{x:d}(x) = d$ , we can equivalently say (with  $\mathcal{M} = (\mathcal{D}, \mathcal{I})$ ):

$\models_{\mathcal{M}} (\forall x \psi) [U]$  iff for all  $d \in \mathcal{D}$ ,  $\models_{\mathcal{M}} \psi [U_{x:d}]$ .

Note: Since  $U_{x:d}(x) = d$  by def<sup>n</sup> of  $U_{x:d}$ ,  
 $T_{IU_{x:d}}(x) = U_{x:d}(x) = d$  by def<sup>n</sup> of  $T_{IU}$

e.g.,  $\mathcal{M} = (\mathcal{D}, \mathcal{I})$   
 $\{a, b\}$   $\rightarrow$   $A^{\mathcal{I}} = a$   
 $B^{\mathcal{I}} = b$   
 $P^{\mathcal{I}} = \{a, b\}$   
 $Q^{\mathcal{I}} = \{a\}$

$U =$  any v.a. we like, for at least variable  $x$

$\models_{\mathcal{M}} (\forall x P(x)) [U] \quad ??$

iff for all  $d \in \mathcal{D}$ ,  $\models_{\mathcal{M}} P(x) [U_{x:d}]$  Justific<sup>n</sup>?

iff for all  $d \in \{a, b\}$ ,  $T_{IU_{x:d}}(x) \in P^{\mathcal{I}} = \{a, b\}$  ...

iff for all  $d \in \{a, b\}$ ,  $U_{x:d}(x) \in \{a, b\}$

iff for all  $d \in \{a, b\}$ ,  $d \in \{a, b\}$

TRUE

Note: it didn't matter what  $U$  is!

e.g.,  $\boxed{F_{\mathcal{M}} (\forall x (Q(x) \Rightarrow P(x))) [U] ??}$

iff for all  $d \in \mathcal{D}$ ,  $F_{\mathcal{M}} (Q(x) \Rightarrow P(x)) [U_{x:d}]$

iff for all  $d \in \mathcal{D}$ , not  $F_{\mathcal{M}} Q(x) [U_{x:d}]$  or  $F_{\mathcal{M}} P(x) [U_{x:d}]$

iff for all  $d \in \{a, b\}$ , not  $d \in Q^I = \{a\}$  or  $d \in P^I = \{a, b\}$

TRUE

Note:  $U$  didn't matter!

(Generally true for closed wffs)

### Satisfaction conditions for $\exists$

$F_{\mathcal{M}} (\exists x \varphi) [U]$  iff for some  $d \in \mathcal{D}$ ,  $F_{\mathcal{M}} \varphi [U_{x:d}]$

e.g.,  $\mathcal{M}$ ,  $U$  as above,

$\boxed{F_{\mathcal{M}} (\exists x Q(x)) [U] ??}$

iff for some  $d \in \mathcal{D}$ ,  $F_{\mathcal{M}} Q(x) [U_{x:d}]$

iff for some  $d \in \{a, b\}$ ,  $\top_{U_{x:d}}(x) \in Q^I = \{a\}$

iff for some  $d \in \{a, b\}$ ,  $d \in \{a\}$  TRUE

(use  $d = a$ )

Now redefine truth (at least for wffs involving variables) in terms of satisfaction:

$\boxed{F_{\mathcal{M}} \varphi \text{ iff for all v.a.'s } U, F_{\mathcal{M}} \varphi [U]}$

e.g., in the previous model, we found

$F_{\mathcal{M}} (\forall x P(x)) [U]$ ,

$F_{\mathcal{M}} (\forall x (Q(x) \Rightarrow P(x))) [U]$ ,

$F_{\mathcal{M}} (\exists x Q(x)) [U]$

in each case independently of the choice of  $U$ . So for this  $\mathcal{M}$  we have

$F_{\mathcal{M}} (\forall x P(x))$

$F_{\mathcal{M}} (\forall x (Q(x) \Rightarrow P(x)))$

$F_{\mathcal{M}} (\exists x Q(x))$