

CSC2/455 Software Analysis and Improvement

Proving Programs Correct

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URCS

Outline

Proofs of Program Correctness

Loop Invariants

Theorem Proving

Postscript

Proofs of Program Correctness

Loop Invariants

Theorem Proving

Postscript

What is a correct program?

- A program that meets its specification is a correct program
- What is the correct specification for a program?
 - The scope of this question is beyond this course
 - Not entirely technical
- Our goal is only to study methods that check if a program meets its provided specification
 - Technical only

Our simple program

```
void fn(int k) {  
    int x = k;  
    int c = 0;  
  
    while(x > 0) {  
        c = c + 1;  
        x = x - 1;  
    }  
  
    assert(x == 0);  
    assert(c == k);  
}
```

Will those assertions always be true? [i.e. are they always valid?]

CBMC: Try #1

```
$ cbmc --function fn --trace simple1.c
CBMC version 5.10 (cbmc-5.10) 64-bit x86_64 linux
Parsing simple1.c
Converting
Type-checking simple1
Generating GOTO Program
Adding CPROVER library (x86_64)
Removal of function pointers and virtual functions
Generic Property Instrumentation
Running with 8 object bits, 56 offset bits (default)
Starting Bounded Model Checking
Unwinding loop fn.0 iteration 1 file simple1.c line 14 function fn thread 0
Unwinding loop fn.0 iteration 2 file simple1.c line 14 function fn thread 0
Unwinding loop fn.0 iteration 3 file simple1.c line 14 function fn thread 0
Unwinding loop fn.0 iteration 4 file simple1.c line 14 function fn thread 0
Unwinding loop fn.0 iteration 5 file simple1.c line 14 function fn thread 0
Unwinding loop fn.0 iteration 6 file simple1.c line 14 function fn thread 0
Unwinding loop fn.0 iteration 7 file simple1.c line 14 function fn thread 0
Unwinding loop fn.0 iteration 8 file simple1.c line 14 function fn thread 0
Unwinding loop fn.0 iteration 9 file simple1.c line 14 function fn thread 0
Unwinding loop fn.0 iteration 10 file simple1.c line 14 function fn thread 0
Unwinding loop fn.0 iteration 11 file simple1.c line 14 function fn thread 0
Unwinding loop fn.0 iteration 12 file simple1.c line 14 function fn thread 0
...
```

Whoops, infinite loop!

CBMC: Try #2

```
$ cbmc --function fn --unwind 10 --trace simple1.c
CBMC version 5.10 (cbmc-5.10) 64-bit x86_64 linux
...
Starting Bounded Model Checking
Unwinding loop fn.0 iteration 1 file simple1.c line 14 function fn thread 0
Unwinding loop fn.0 iteration 2 file simple1.c line 14 function fn thread 0
...
Unwinding loop fn.0 iteration 8 file simple1.c line 14 function fn thread 0
Unwinding loop fn.0 iteration 9 file simple1.c line 14 function fn thread 0
Not unwinding loop fn.0 iteration 10 file simple1.c line 14 function fn thread 0
...

** Results:
[fn.assertion.1] assertion x == 0: FAILURE
[fn.assertion.2] assertion c == k: FAILURE

...

k: -2147483648 (10000000 00000000 00000000 00000000)
x:-2147483648 (10000000 00000000 00000000 00000000)
c=0 (00000000 00000000 00000000 00000000)

Violated property: assertion x == 0
x == 0

Violated property: assertion c == k
c == k
```

- if k is negative (note: output is reformatted to fit)
 - x will not be zero
 - c will not be equal k

Specifying k must always be greater than zero

- We check our specifications, and notice that `fn` should only work on non-negative k

```
void fn(int k) {  
    __CPROVER_assume(k >= 0);  
    ...  
}
```


CBMC: Try #3

```
$ cbmc --unwinding-assertions --function fn --unwind 10 --trace simple1.c
CBMC version 5.10 (cbmc-5.10) 64-bit x86_64 linux
...
Unwinding loop fn.0 iteration 9 file simple1.c line 14 function fn thread 0
Not unwinding loop fn.0 iteration 10 file simple1.c line 14 function fn thread 0
...
** Results:
[fn.assertion.1] assertion x == 0: SUCCESS
[fn.assertion.2] assertion c == k: SUCCESS
[fn.unwind.0] unwinding assertion loop 0: FAILURE

Trace for fn.unwind.0:

INPUT k: 12 (00000000 00000000 00000000 00001100)
c=10 (00000000 00000000 00000000 00001010)
x=2 (00000000 00000000 00000000 00000010)

Violated property:
  unwinding assertion loop 0
```

- CBMC can't show loop terminates for a (fixed) finite number of unwindings
 - Here unwind=10 and CBMC says more unwindings would be needed for $k = 12$
- Conclusions may be unsound

Try out all possible unwindings

- For C , k is still an integer.
 - Finite number of values
 - Could try out all possible unwindings by fixing an upper bound
- Might be feasible for `simple`
 - But add more loops, and time/space increases significantly
- Strategy not even feasible for languages like Python
 - Python has infinite precision integers
- Can we try something else?

```
void fn(int k) {
    int x = k;
    int c = 0;

    while(x > 0) {
        c = c + 1;
        x = x - 1;
    }

    assert(x == 0);
    assert(c == k);
}

int main(void) {
    int k;

    klee_make_symbolic(&k, sizeof(k), "k");
    klee_assume(k >= 0);
    fn(k);
}
```

KLEE, contd.

```
clang -I ~/ext/klee-2.1/include/ -emit-llvm -c -g -O0 -Xclang -disa  
~/ext/klee-2.1/build/bin/klee simple1.bc  
KLEE: output directory is "src/klee-out-0"  
KLEE: Using Z3 solver backend  
^CKLEE: ctrl-c detected, requesting interpreter to halt.  
KLEE: halting execution, dumping remaining states  
  
KLEE: done: total instructions = 5174  
KLEE: done: completed paths = 273  
KLEE: done: generated tests = 273
```

- Symbolic execution using KLEE doesn't seem to work either
 - I interrupted after a minute or so.
 - Without `klee_assume`, KLEE also detects the assertion failure of `x == 0`

What about abstract interpretation?

```
x := k;  
c := 0;  
while(x > 0) {  
    x := (x - 1);  
    c := (c + 1)  
}
```

- Input:
 - $M^\# = \{k \mapsto [0, +\infty), x \mapsto \top, c \mapsto \top\}$
- Output:
 - $\{ 'k': (0, +\text{inf}), 'x': (0, 0), 'c': (0, +\text{inf}) \}$
 - $M^\# = \{k \mapsto [0, +\infty), x \mapsto [0, 0], c \mapsto [0, +\infty)\}$

Does M^\sharp allow us to prove our assertions?

$$M^\sharp = \{k \mapsto [0, +\infty), x \mapsto [0, 0], c \mapsto [0, +\infty)\}$$

- Logically $P : (k \geq 0) \wedge (x = 0) \wedge (c \geq 0)$
 - We want to prove $a_0 : x = 0$
 - We want to prove $a_1 : c = k$
- For a_0
 - If P is valid, then so is a subset of P , in particular $P_0 : (x = 0)$
 - (This is because $a \wedge b \wedge c \implies a$ is valid)
 - $P_0 \implies a_0$ is valid (also written as $P_0 \models a_0$)
- This won't work for a_1
 - $P_1 : (k \geq 0) \wedge (c \geq 0)$ [any subset can be chosen]
 - $P_1 \not\models (c = k)$
 - Not strong enough. Counterexample: $k = 6, c = 5$
 - Recall intervals domain is not relational, so can't relate c to k

simple.c, logically deriving P_0

```
void fn(int k) {
    int x = k;
    int c = 0;

    while(x > 0) {
        c = c + 1;
        x = x - 1;
    }

    assert(x == 0);
    assert(c == k);
}
```

- Clearly, P_0 captures the state of the program at the end of the loop well enough to allow us to prove $x = 0$
- Can we derive P_0 (logically)?
 - First glance, *only* from loop condition, all we can say is that $x \leq 0$ if loop executes and exits.
 - Not strong enough to prove $x = 0$

Loops ...

- Loops may execute zero, a finite number, or an infinite number of iterations
 - Bounded Model Checkers: Can't handle loops soundly without a fixed upper bound
 - Symbolic checkers: same
 - Abstract interpretation: Approximation may prevent us from verifying some properties
- But if we can find a P that captures the state of the program at the end of a loop
 - executing zero, finite or infinite number of iterations
 - P may be strong enough to prove properties we're interested in
 - without having to model the loop iteration by iteration

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Loop Invariants

- A loop invariant is a condition over the program state that holds:
 - Before the loop
 - At the beginning of each iteration
 - At the end of each iteration

A loop invariant in `simple.c`

```
assert(x >= 0);  
  
while(x > 0) {  
    assert(x >= 0);  
  
    c = c + 1;  
    x = x - 1;  
  
    assert(x >= 0);  
}
```

- $x \geq 0$ holds before the loop (since $x = k$, and $k \geq 0$)
- $x \geq 0$ holds at beginning of iteration, since $x > 0$ (from loop condition)
- $x \geq 0$ holds at end of iteration
 - x is reduced by 1 each iteration
 - $x > 0 \implies x \geq 1 \implies x - 1 \geq 0$

Using the loop invariant to prove $x == 0$

- At end of loop
 - $x \leq 0$ (from loop condition, if loop exits, then $\neg(x > 0)$ holds)
 - $x \geq 0$ (from loop invariant)
 - $x \leq 0 \wedge x \geq 0 \implies x = 0$
- What about $c == k$?

Trying out some candidate loop invariants for $c == k$

- Will $c \leq k$ work?

```
assert(c <= k);  
while(x > 0) {  
    assert(c <= k);  
  
    c = c + 1;  
    x = x - 1;  
  
    assert(c <= k);  
}
```

- Definitely holds before loop ($k \geq 0$, and $c = 0$)
- But harder to show that c won't exceed k during loop
 - We *know* it is true, but hard to prove!
 - We only know $x > 0$ at the beginning of each iteration
 - Hard to show that $c + 1 \leq k$ from that premise (even assuming $c \leq k$)
 - In fact $c \leq k$ allows $c = k$ which would mean $c + 1 > k$!

Change the loop condition?

```
assert(c <= k);
while(c < k) {
    assert(c <= k);

    c = c + 1;
    x = x - 1;

    assert(c <= k);
}
```

- Definitely holds before loop ($k \geq 0$, and $c = 0$)
- Holds on entry to loop as well $c < k \implies c \leq k$
- Holds after each iteration as well:
 - $c + 1 \leq k + 1$, (from invariant)
 - $c < k$ (from loop condition)
 - $c + 1 \leq k$

Using the loop invariant to prove $c == k$

- At end of loop
 - $c \geq k$ (from loop condition, if loop exits, then $\neg(c < k)$ holds)
 - $c \leq k$ (from loop invariant)
 - $c \leq k \wedge c \geq k \implies c = k$
- What about $x == 0$?
 - Back to square one?
- How about combining the loop conditions and the invariants?

Combining the loop invariants and loop conditions

```
assert(x >= 0 && c <= k);
while(x > 0 && c < k) {
    assert(x >= 0 && c <= k);

    c = c + 1;
    x = x - 1;

    assert(x >= 0 && c <= k);
}
```

- This doesn't seem to work
 - Not strong enough to imply either assertion after combination with loop exit condition!
 - If you work it out, you may be tempted to change the loop condition...

Let's look at some concrete program executions

- $k = 5$

entry : k: 5, x: 5, c: 0

end: k: 5, x: 4, c: 1

entry: k: 5, x: 4, c: 1

end: k: 5, x: 3, c: 2

entry: k: 5, x: 3, c: 2

end: k: 5, x: 2, c: 3

entry: k: 5, x: 2, c: 3

end: k: 5, x: 1, c: 4

entry: k: 5, x: 1, c: 4

end: k: 5, x: 0, c: 5

exit: k: 5, x: 0, c: 5

- Do you see a relation between x , c , and k ?
- Do you see a pattern that is unchanging (i.e. invariant)?

Invariant candidate #4: $x + c == k$

```
assert(x + c == k);
while(x > 0) {          /* note original loop condition */
    assert(x + c == k);

    c = c + 1;
    x = x - 1;

    assert(x + c == k);
}
```

- Clearly holds before entering loop and on first iteration
 - $x = k \wedge c = 0 \implies x + c = k$
- Assume holds at some iteration
 - $x + c = k$
- Then, it still holds at end of iteration (and next iteration)
 - $x - 1 + c + 1 = k$
- (Inductive argument)

Proving a_0 and a_1

- $P : \neg(x > 0) \wedge (x + c = k)$
 - For a_0 : $(x \leq 0) \wedge (x + c = k) \implies x = 0$
 - For a_1 : $(x \leq 0) \wedge (x + c = k) \implies c = k$
- Can't prove these using P as derived, since P admits $x < 0$.
 - We want $x = 0$ for the proof to go through
 - Without $x = 0$, setting $x = -1$ is a counterexample for both $P \implies a_0$ and $P \implies a_1$
- But we can derive that $x \geq 0$
 - We are given that $k \geq 0$
 - $\{k \geq 0\} x := k \{x \geq 0\}$ (*assignment axiom*)
- We can therefore strengthen P by adding $x \geq 0$
 - Allowed since $P \wedge \text{true}$ is still true
- This allows both the proofs to go through!
 - $(x \leq 0) \wedge (x + c = k) \wedge (x \geq 0) \implies x = 0$

Summary

- A loop invariant captures the effects of a loop on the program state
 - Without having to “run” or approximate states
 - Just have to prove the invariant satisfies the definition
- A useful loop invariant allows us to prove properties
 - May require additional facts given or derived from other parts of the program
- How to find loop invariants?
 - Use the Feynman “Algorithm” [not serious]
 - No general technique to find loop invariants!

Partial Correctness

- Big elephant in the room
 - Our proofs only hold if the loops terminate!
- Do the loops in the programs so far terminate?
 - Easy to show that they do
 - All have a strictly decreasing variable
 - Loop terminates when that variable reaches zero
- But, revisit the loop condition $x > 0 \ \&\& \ c < k$
 - with loop invariant $x \geq 0 \wedge c \leq k$
 - the negation of the loop condition prevents us from proving a_0 and a_1
- Might be tempted to use $x > 0 \ || \ c < k$ after figuring that out

Example

```
void fn(int k) {
    int x = k;
    int c = 0;

    assert(x >= 0 && c <= k);

    while(x > 0 || c < k) {
        assert(x >= 0 && c <= k);
        c = c + 1;
        x = x - 1;
        assert(x >= 0 && c <= k);
    }

    assert(x == 0);
    assert(c == k);
}
```

- Assume the loop invariant is still valid
- How do you prove the loop terminates?
 - Need to show that x becomes zero at the same time as c becomes k

Total Correctness

Total correctness = Partial Correctness + Termination

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Interactive Theorem Provers

- Sometimes called Proof Assistants
 - Isabelle
 - Coq
 - Lean
- Allow you to write proofs
 - Assist you in solving them
 - Proof writing is undecidable in general
- Verify your proofs are correct!
- Actually make writing proofs fun
 - Though still very tedious?





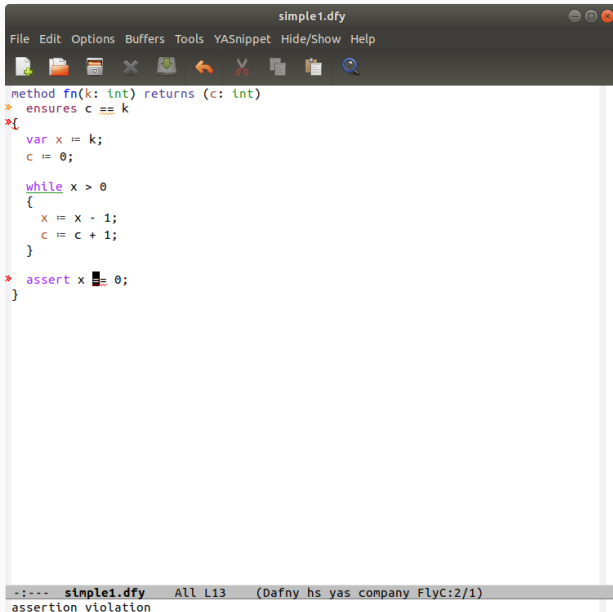
```
method DutchFlag(a: array<Color>)
  requires a ≠ null modifies a
  > ensures ∀ i,j · 0 ≤ i < j < a.Length ⇒ Ordered(a[i], a[j])
  ensures multiset(a[..]) == old(multiset(a[..]))
{
  var r, w, b = 0, 0, a.Length;
  > while w ≠ b
    invariant 0 ≤ r ≤ w ≤ b ≤ a.Length;
    invariant ∀ i · 0 ≤ i < r ⇒ a[i] == Red
    invariant multiset(a[..]) == old(multiset(a[..]))
  {
    match a[w]
    case Red ⇒
      a[r], a[w] = a[w], a[r];
      r, w = r + 1, w + 1;
    case White ⇒
      w = w + 1;
    case Blue ⇒
      b = b - 1;
  }
```

- A free programming language that only compiles programs that can be verified
 - Generates C#, JS or Go
- Termed “auto-active program verifier”
 - Can verify your programs as you type them
- Can be used as a batch style compiler
 - But best used with an Editor
 - VS Code and Emacs supported

Obtaining Dafny

- Available for free for Windows, Linux and macOS
 - Even in Debian/Ubuntu repository
 - (though old version)
- I'm using the version from the Github repo

Simple program in Dafny



The screenshot shows a window titled "simple1.dfy" with a menu bar (File, Edit, Options, Buffers, Tools, YASnippet, Hide/Show, Help) and a toolbar. The code in the editor is as follows:

```
method fn(k: int) returns (c: int)
  > ensures c == k
  >{
    var x := k;
    c := 0;

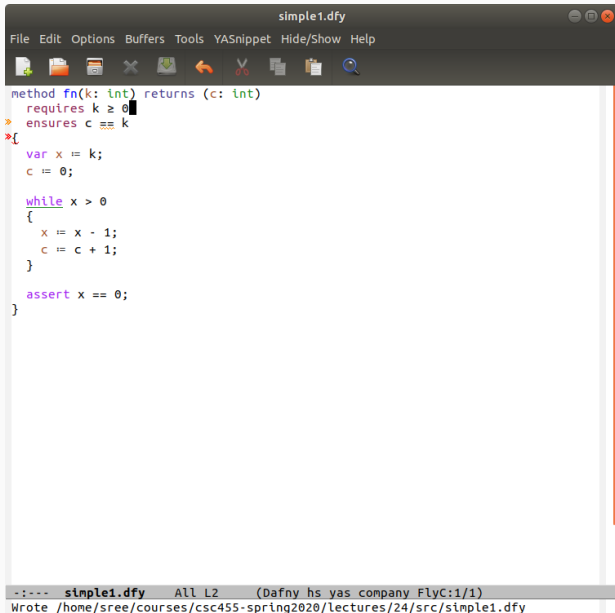
    while x > 0
    {
      x := x - 1;
      c := c + 1;
    }

    > assert x == 0;
  }
```

At the bottom of the window, a status bar displays the following information:

```
-:--- simple1.dfy All L13 (Dafny hs yas company FlyC:2/1)
assertion violation
```

Adding requires



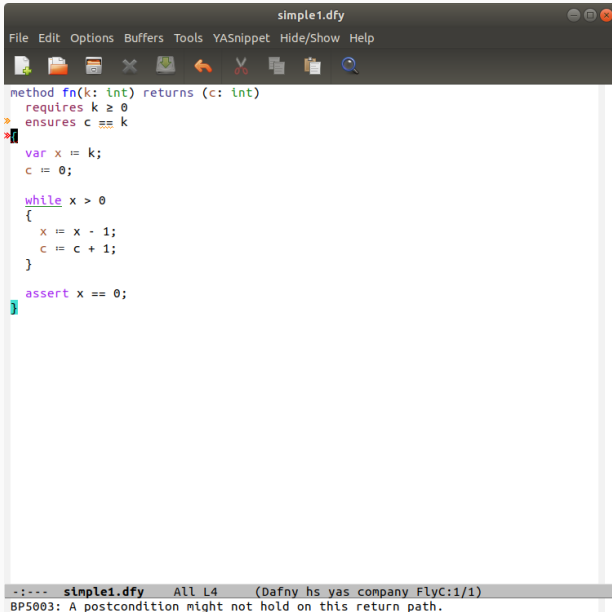
```
simple1.dfy
File Edit Options Buffers Tools YASnippet Hide/Show Help
method fn(k: int) returns (c: int)
  requires k ≥ 0
  ensures c == k
{
  var x := k;
  c := 0;

  while x > 0
  {
    x := x - 1;
    c := c + 1;
  }

  assert x == 0;
}
```

--- simple1.dfy All L2 (Dafny hs yas company FlyC:1/1)
Wrote /home/sree/courses/csc455-spring2020/lectures/24/src/simple1.dfy

Postcondition might not hold



```
simple1.dfy
File Edit Options Buffers Tools YASnippet Hide/Show Help
method fn(k: int) returns (c: int)
  requires k ≥ 0
  ensures c == k
{
  var x := k;
  c := 0;

  while x > 0
  {
    x := x - 1;
    c := c + 1;
  }

  assert x == 0;
}
```

--- simple1.dfy All L4 (Dafny hs yas company FlyC:1/1)
BP5003: A postcondition might not hold on this return path.

Add $x \geq 0$ invariant

```
simple1.dfy
File Edit Options Buffers Tools YASnippet Hide/Show Help
method fn(k: int) returns (c: int)
  requires k ≥ 0
  ensures c == k
x{
  var x := k;
  c := 0;

  while x > 0
    invariant x ≥ 0
    {
      x := x - 1;
      c := c + 1;
    }

  assert x == 0;
}

-:***- simple1.dfy All L4 (Dafny hs yas company FlyC:1/1)
BP5003: A postcondition might not hold on this return path.
```

Add $c \leq k$ invariant

```
simple1.dfy
File Edit Options Buffers Tools YASnippet Hide/Show Help
method fn(k: int) returns (c: int)
  requires k ≥ 0
  > ensures c == k
  >{
    var x := k;
    c := 0;

    while x > 0
    > invariant c ≤ k
    {
      x := x - 1;
      c := c + 1;
    }

    assert x == 0;
  }
}

-:--- simple1.dfy All L9 (Dafny hs yas company FlyC:2/1)
BP5005: This loop invariant might not be maintained by the loop.
```


Add $c \leq k$ invariant, contd

```
simple1.dfy
File Edit Options Buffers Tools YASnippet Hide/Show Help
method fn(k: int) returns (c: int)
  requires k ≥ 0
  ensures c == k
{
  var x := k;
  c := 0;

  while c < k
    invariant c ≤ k
    {
      x := x - 1;
      c := c + 1;
    }
  > assert x == 0;
}
```

-.**-* simple1.dfy All L15 (Dafny hs yas company FlyC:1/0)
assertion violation

Combining invariants

```
simple1.dfy
File Edit Options Buffers Tools YASnippet Hide/Show Help
method fn(k: int) returns (c: int)
  requires k ≥ 0
  > ensures c == k
  >{
    var x := k;
    c := 0;

    while x > 0 ∧ c < k
      invariant x ≥ 0 ∧ c ≤ k
      {
        x := x - 1;
        c := c + 1;
      }
  }
  > assert x == 0;
}
```

-.**-* simple1.dfy All L15 (Dafny hs yas company FlyC:2/1)
assertion violation

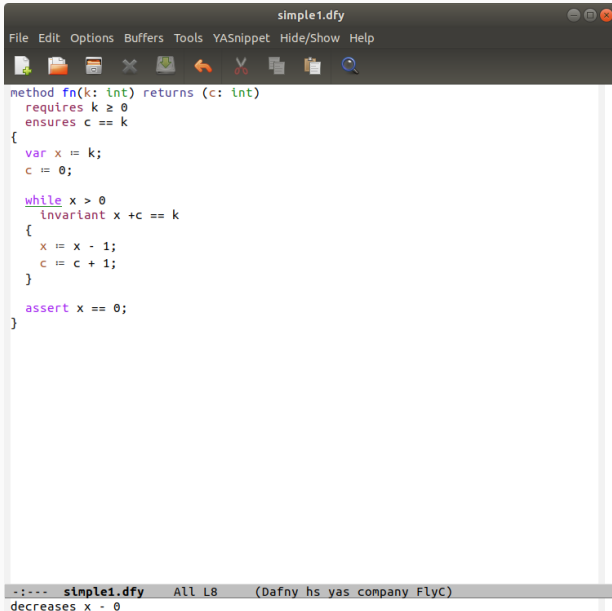
Combining invariants - Part #2

```
simple1.dfy
File Edit Options Buffers Tools YASnippet Hide/Show Help
method fn(k: int) returns (c: int)
  requires k ≥ 0
  > ensures c == k
  x|
  var x := k;
  c := 0;

  while x > 0 ∧ c < k
    invariant x ≥ 0 ∧ c ≤ k
    {
      x := x - 1;
      c := c + 1;
    }
  > assert x == 0;
  |

-:***- simple1.dfy All L4 (Dafny hs yas company FlyC:2/1)
BP5003: A postcondition might not hold on this return path.
```

Final Invariant



The screenshot shows a window titled "simple1.dfy" with a menu bar (File, Edit, Options, Buffers, Tools, YASnippet, Hide/Show, Help) and a toolbar. The code in the editor is as follows:

```
method fn(k: int) returns (c: int)
  requires k ≥ 0
  ensures c == k
{
  var x := k;
  c := 0;

  while x > 0
    invariant x + c == k
  {
    x := x - 1;
    c := c + 1;
  }

  assert x == 0;
}
```

At the bottom of the window, a status bar displays: `-- simple1.dfy All L8 (Dafny hs yas company FlyC) decreases x - 0`

Non-terminating loop

```
simple1.dfy
File Edit Options Buffers Tools YASnippet Hide/Show Help
method fn(k: int) returns (c: int)
  requires k ≥ 0
  ensures c == k
{
  var x := k;
  c := 0;
  > while x > 0 ∨ c < k
    invariant x ≥ 0 ∧ c ≤ k
    {
      x := x - 1;
      c := c + 1;
    }
  assert x == 0;
}

-:***- simple1.dfy All L8 (Dafny hs yas company FlyC:1/0)
decreases
cannot prove termination; try supplying a decreases clause for the loop
```

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Further Resources

- We focused entirely on loop invariants today
 - Their utility and ability to model entire loop executions
 - Their use in proving properties
- Introduced you to Dafny
 - The Dafny Project at Microsoft Research
 - More reading (including 4-part video lectures)
- Next week: Hoare Logic
 - Source of the assignment axiom, and other rules for deriving program facts
 - Strongly recommend reading Background Reading on Hoare Logic, by Mike Gordon

Homework

- Let $\text{popcount}(x)$ be the number of bits set to 1 in x
- Show that $\text{popcount}(x) - \text{popcount}(x \& (x - 1)) = 1$
 - (where $\&$ is bitwise and)
- Example:
 - 5 is 0b101, 4 is 0b100, $5 \& 4 = 0b100 = 4$