

Lecture 3: Fourier Analysis

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CSC 259/459, Fall 2024
Computer Imaging & Graphics

Logistics

- Written assignment 1 is up and is due Sept. 11 11:30 AM.
- You can work in groups of 2.
- Course schedule: <https://cs.rochester.edu/courses/259/fall2024/schedule.html>. You will find reading assignments and slides.
 - Some readings are on Blackboard
- Start thinking and talking to me about your final project idea.

The Roadmap

Theoretical Preliminaries

Human Visual Systems

Color in Nature, Arts, & Tech
(a.k.a., the birth, life, and death of light)

Digital Camera Imaging

Modeling and Rendering

Applications



Geometric Transformations

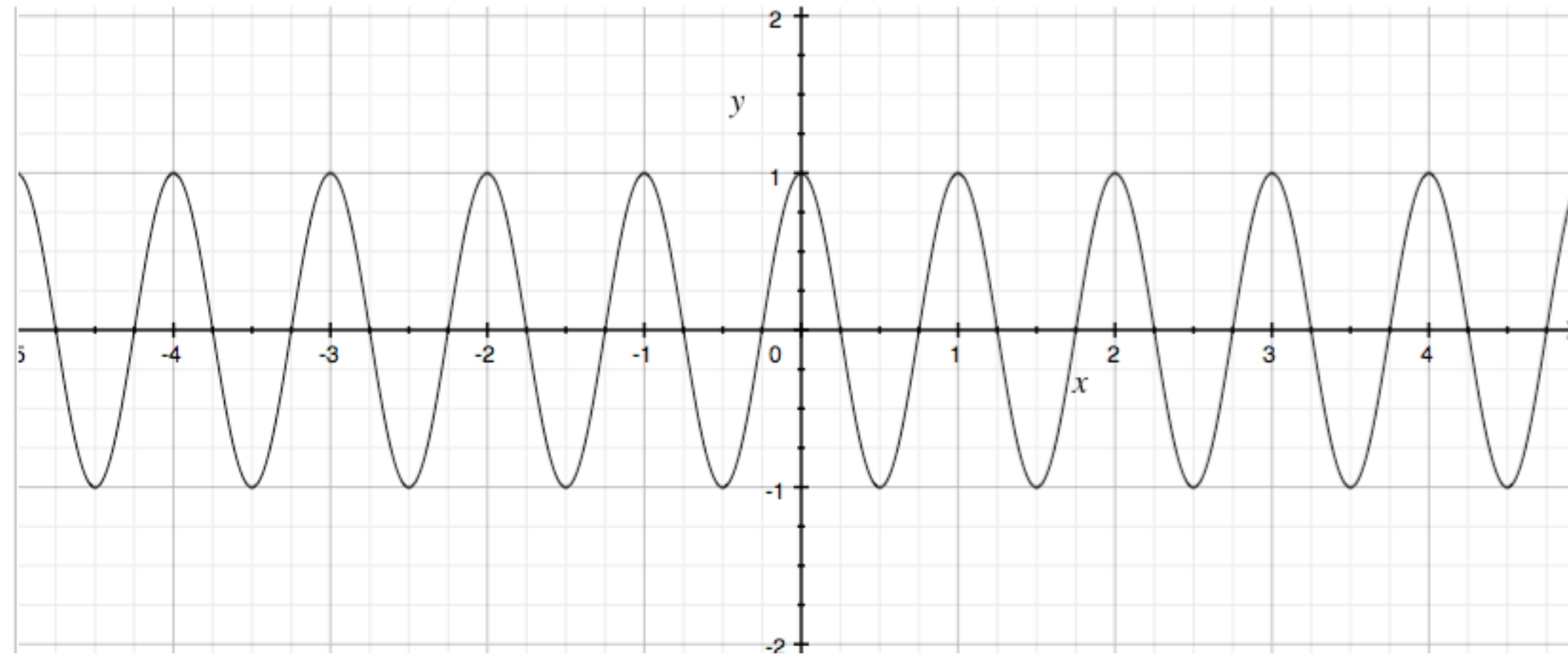
Fourier Series & Transforms

Sampling & Reconstruction

Building Intuitions

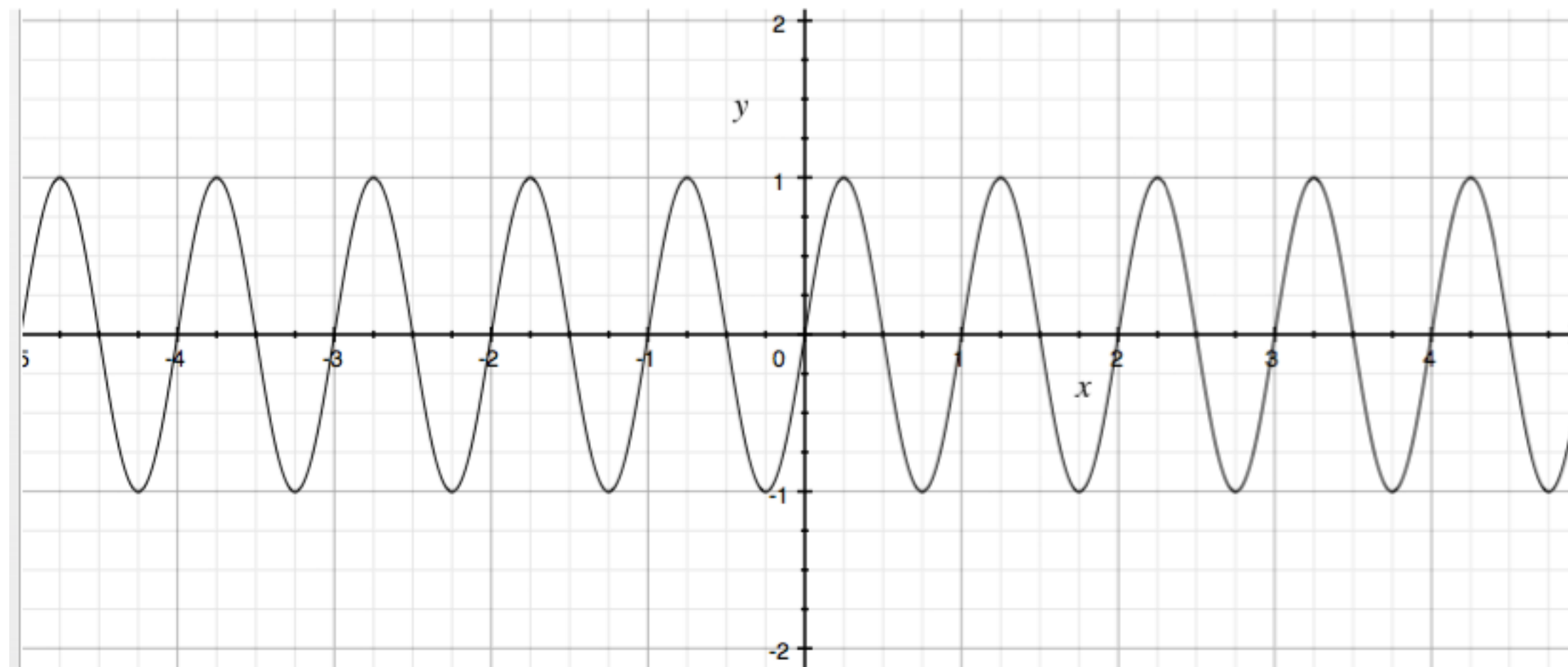
(1D)

Sinusoidal Function



$$\cos 2\pi x$$

Repeating at an interval of 1,
i.e., period (T) = 1
"Even" function

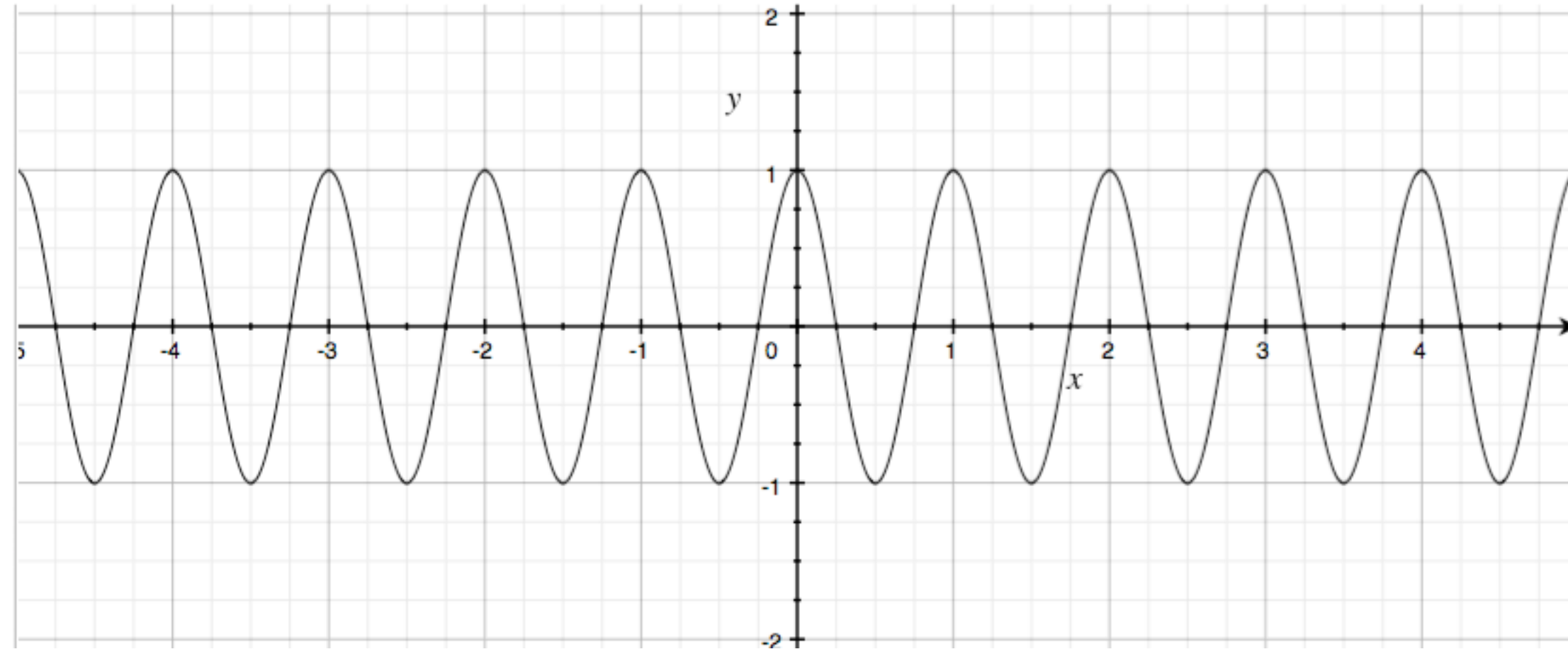


$$\sin 2\pi x$$

Repeating at an interval of 1,
i.e., period (T) = 1
"Odd" function

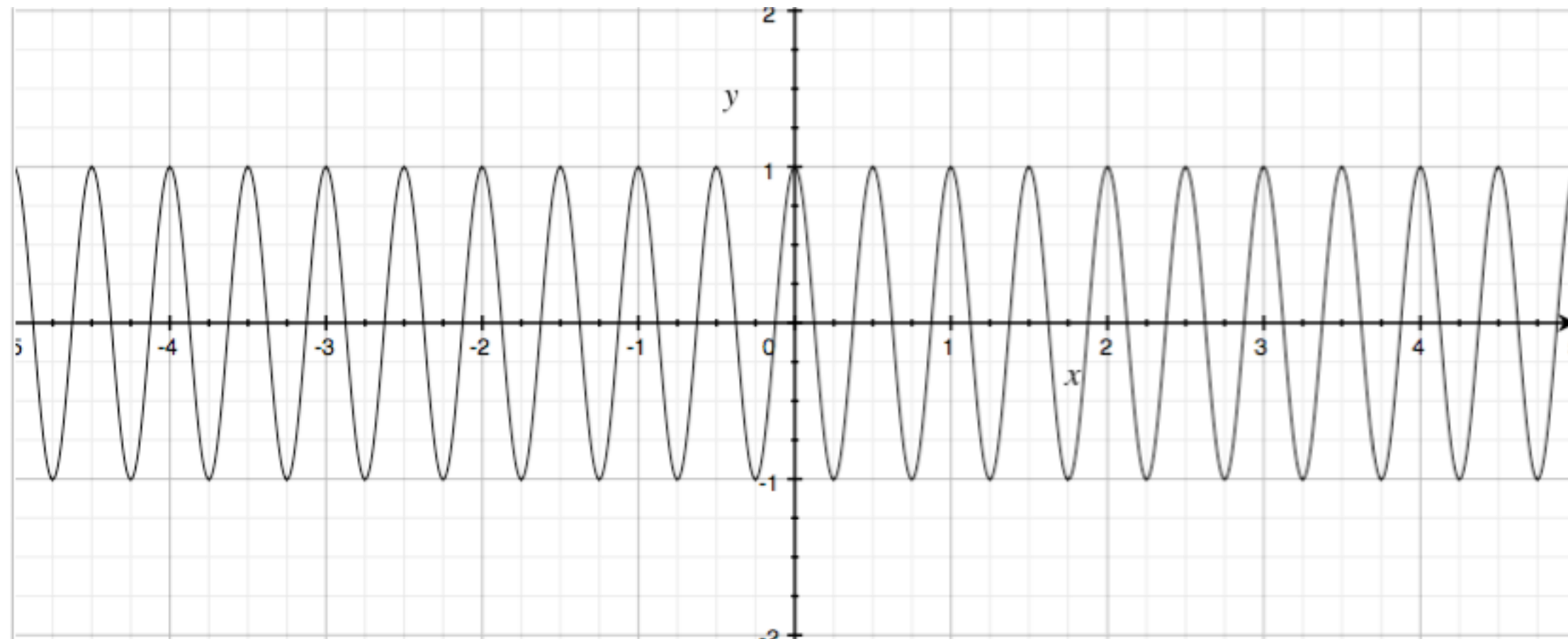
Sinusoidal Function Frequency

$$f = \frac{1}{T}$$



$\cos 2\pi x$

$$f = 1$$



$\cos 4\pi x$

$$f = 2$$

$$\sin kx, \cos kx$$

$$f_{\text{ordinary}} = \frac{k}{2\pi}$$

$$f_{\text{angular}} = k$$

$$T = \frac{2\pi}{k}$$

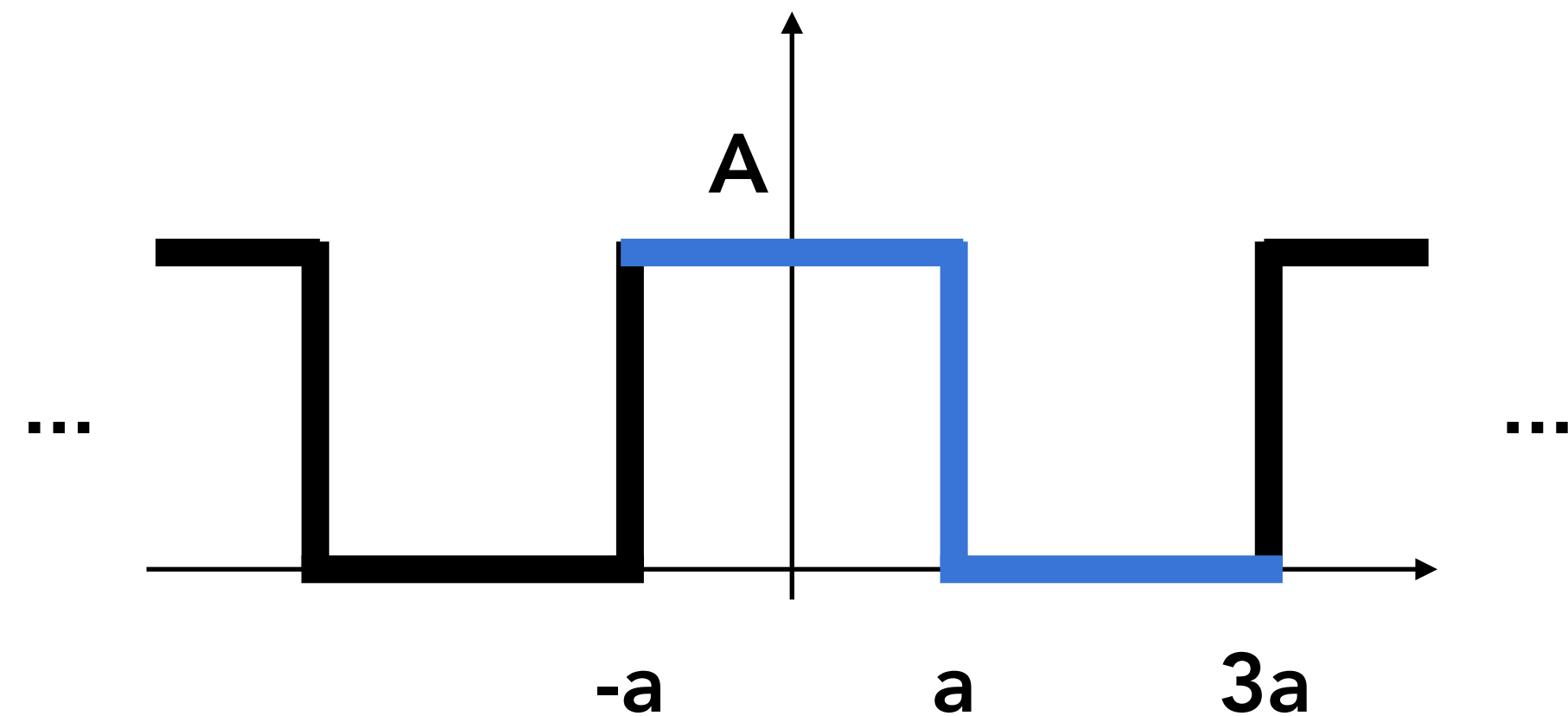
Fourier Series: What It Says



Joseph Fourier (1768 – 1830)

Any **periodic** function can be represented as an **infinite** sum of weighted sines and cosines with **increasing frequencies**.

Two Examples to Build Intuition

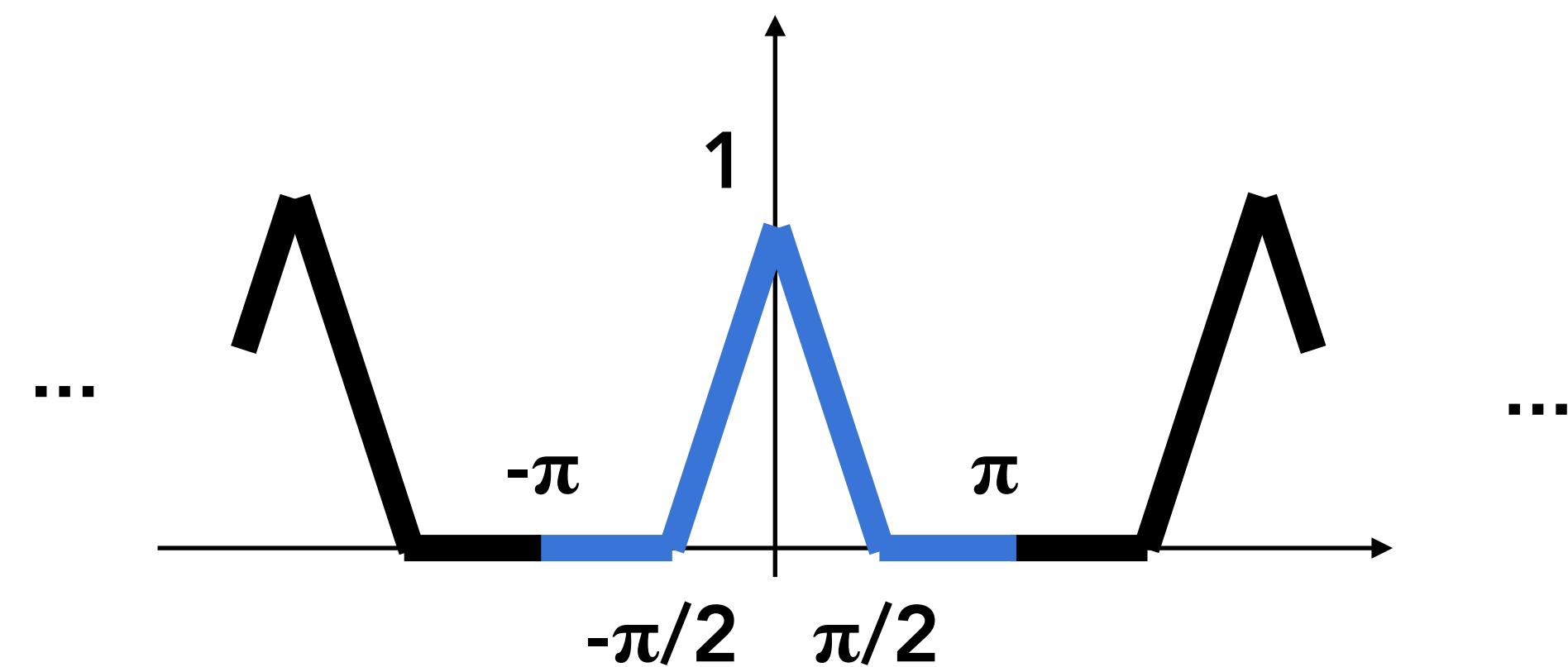


A square function

Period $T = 4a$

Frequency is $1/4a$

Amplitude is A



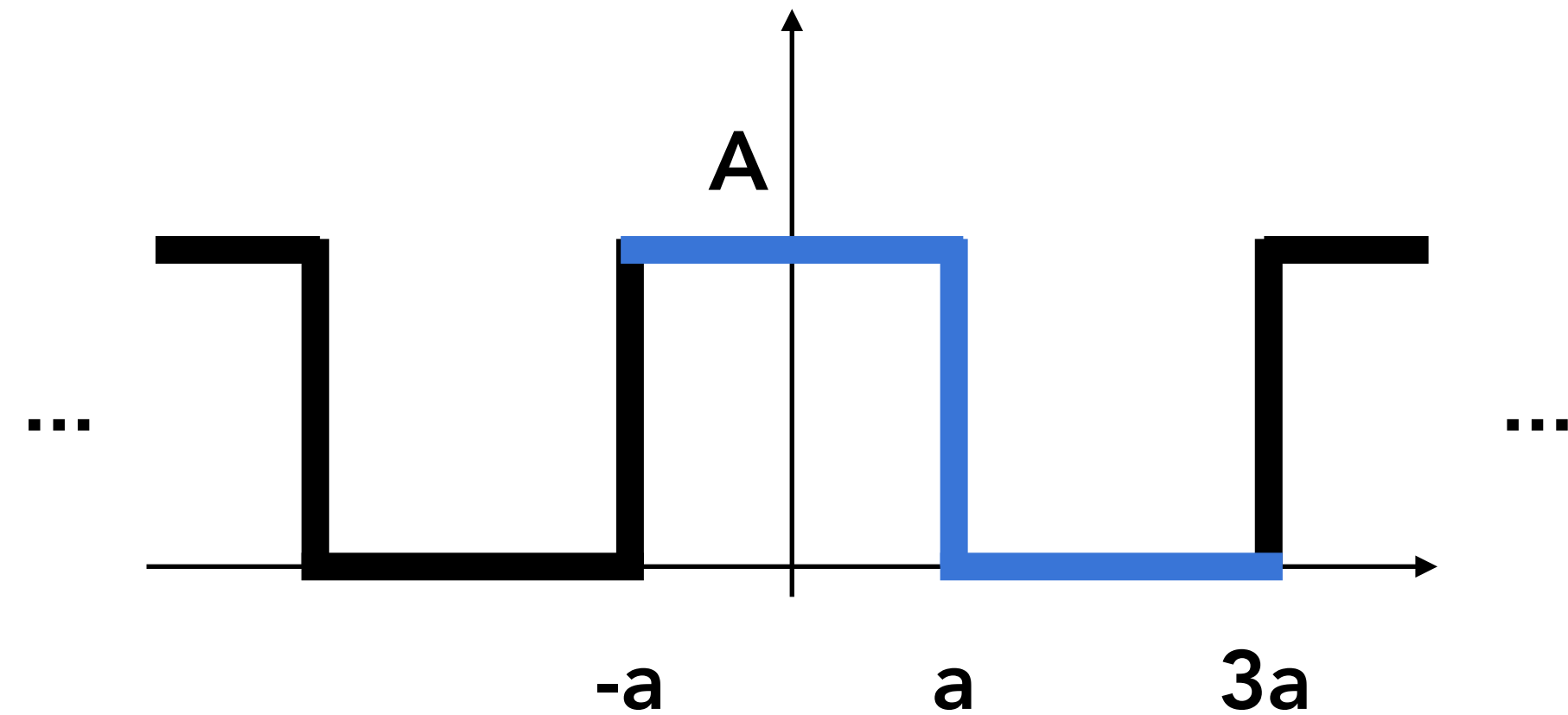
A hat function

Period $T = 2\pi$

Frequency is $1/2\pi$

Amplitude is 1

First Example



A square function

Period $T = 4a$

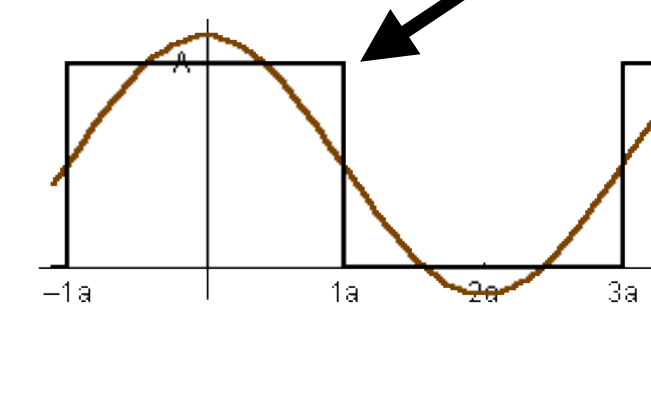
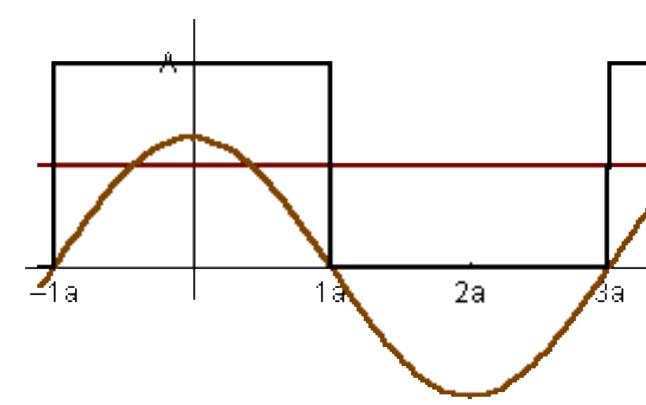
Frequency is $1/4a$

Amplitude is A

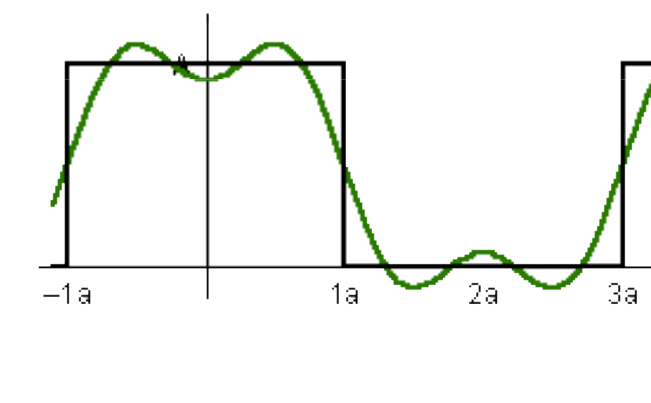
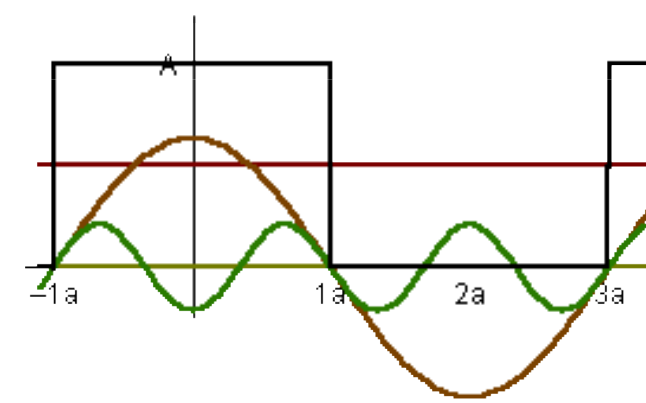
Weights/Coefficients

$$f(x) = \frac{A}{2} + \frac{2A}{\pi} \cos\left(\frac{1}{4a}x\right) + \frac{-2A}{3\pi} \cos\left(\frac{3}{4a}x\right) + \frac{2A}{5\pi} \cos\left(\frac{5}{4a}x\right) + \frac{-2A}{7\pi} \cos\left(\frac{7}{4a}x\right) + \dots$$

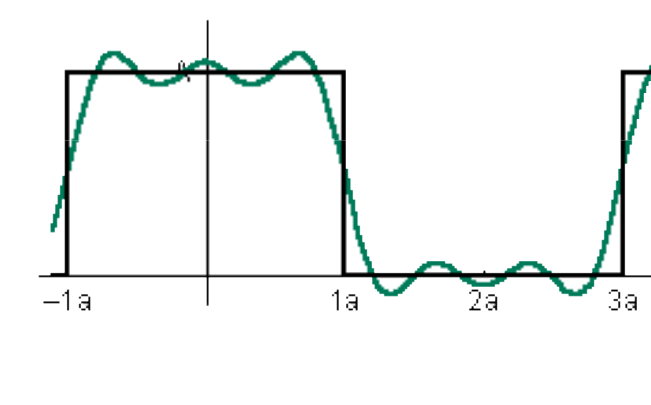
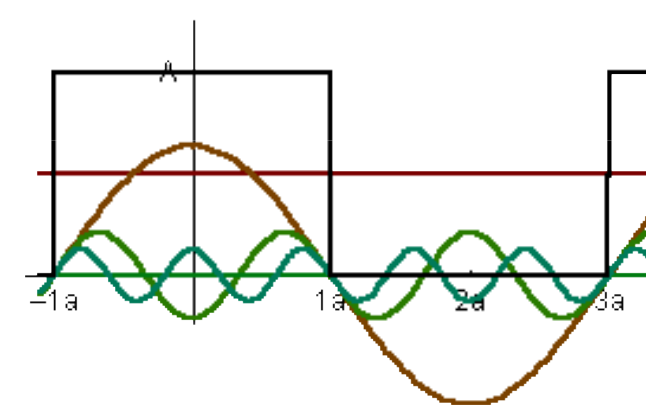
The periodic square function to be represented



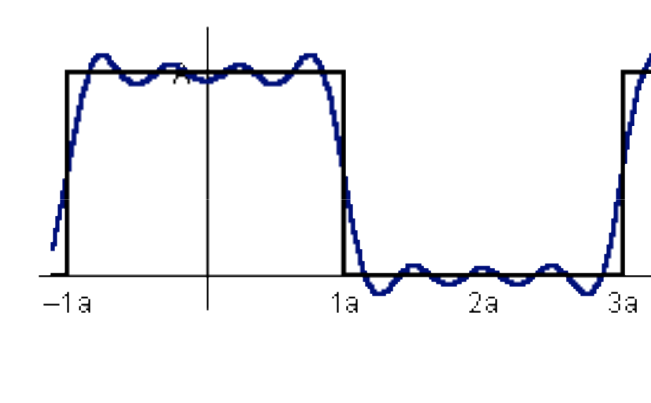
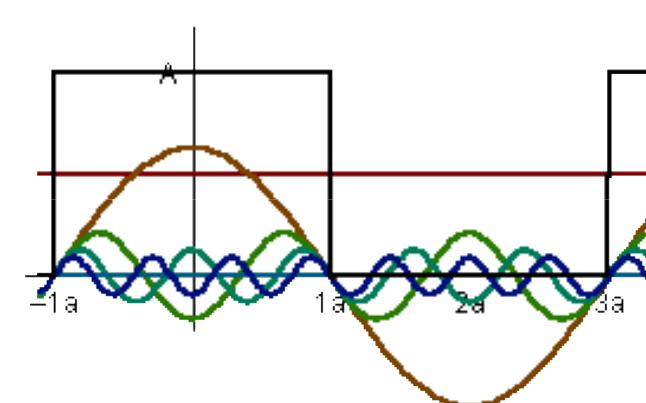
Add first 2 waves



Add first 3 waves

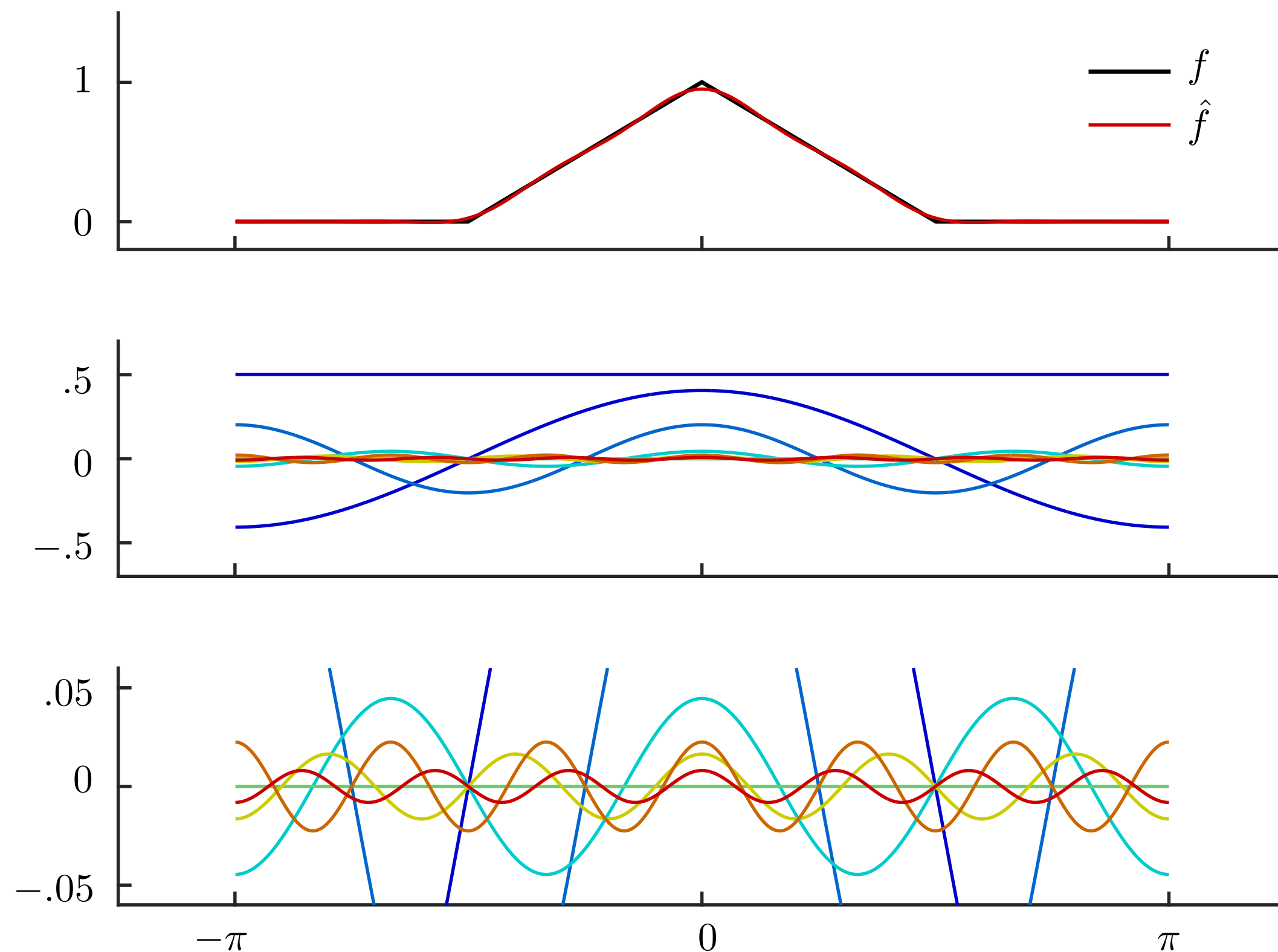


Add first 4 waves



Add first 5 waves

Second Example



The periodic hat function and the sum of the first seven terms

The first seven terms

Zoom in so you see the high-frequency part

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2} \cos(x) + \frac{4}{(3\pi)^2} \cos(3x) + \frac{4}{(5\pi)^2} \cos(5x) + \frac{4}{(7\pi)^2} \cos(7x) + \dots$$

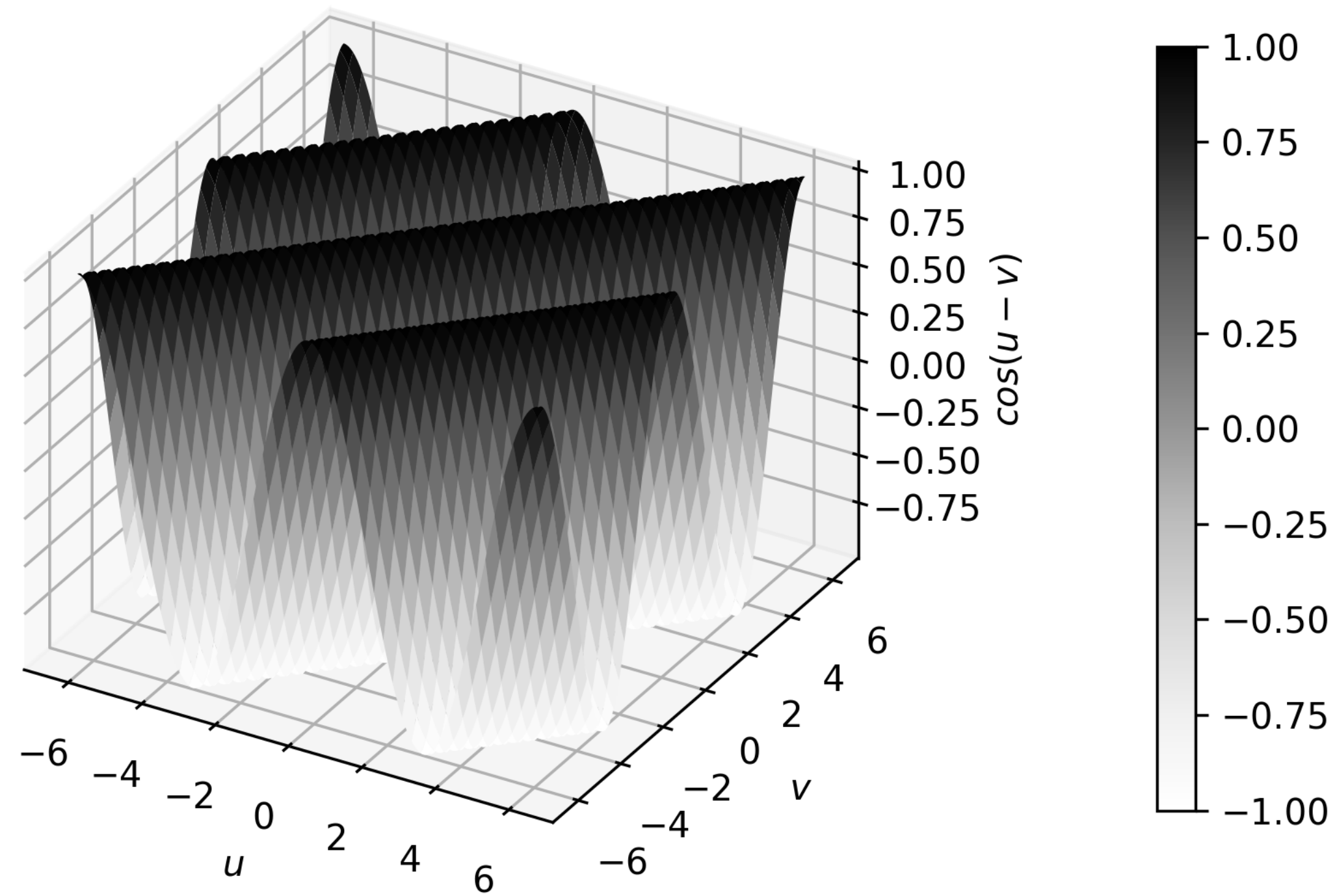
Weights/Coefficients

Building Intuitions

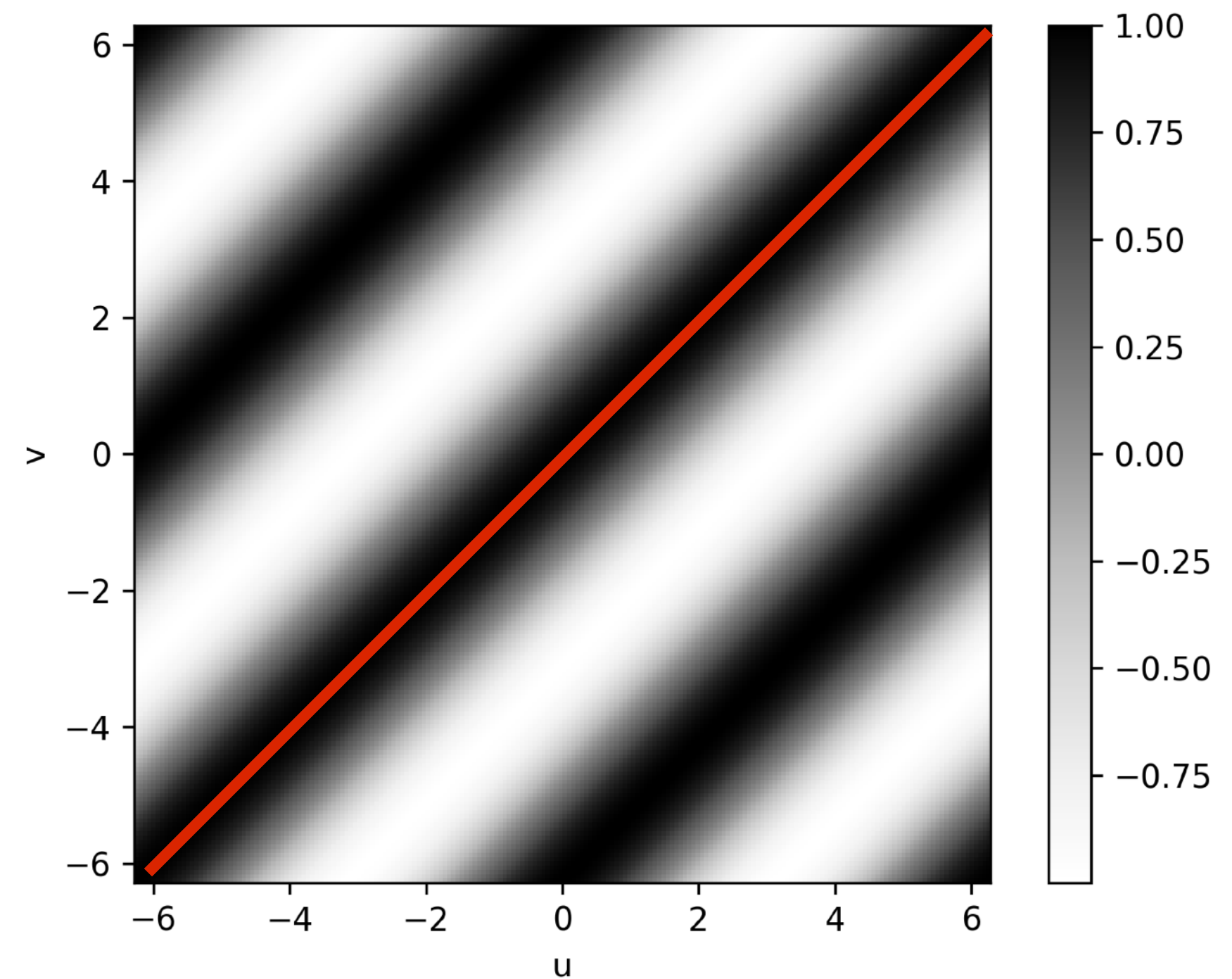
(2D)

2D Sinusoidal Functions

$\cos(u-v)$

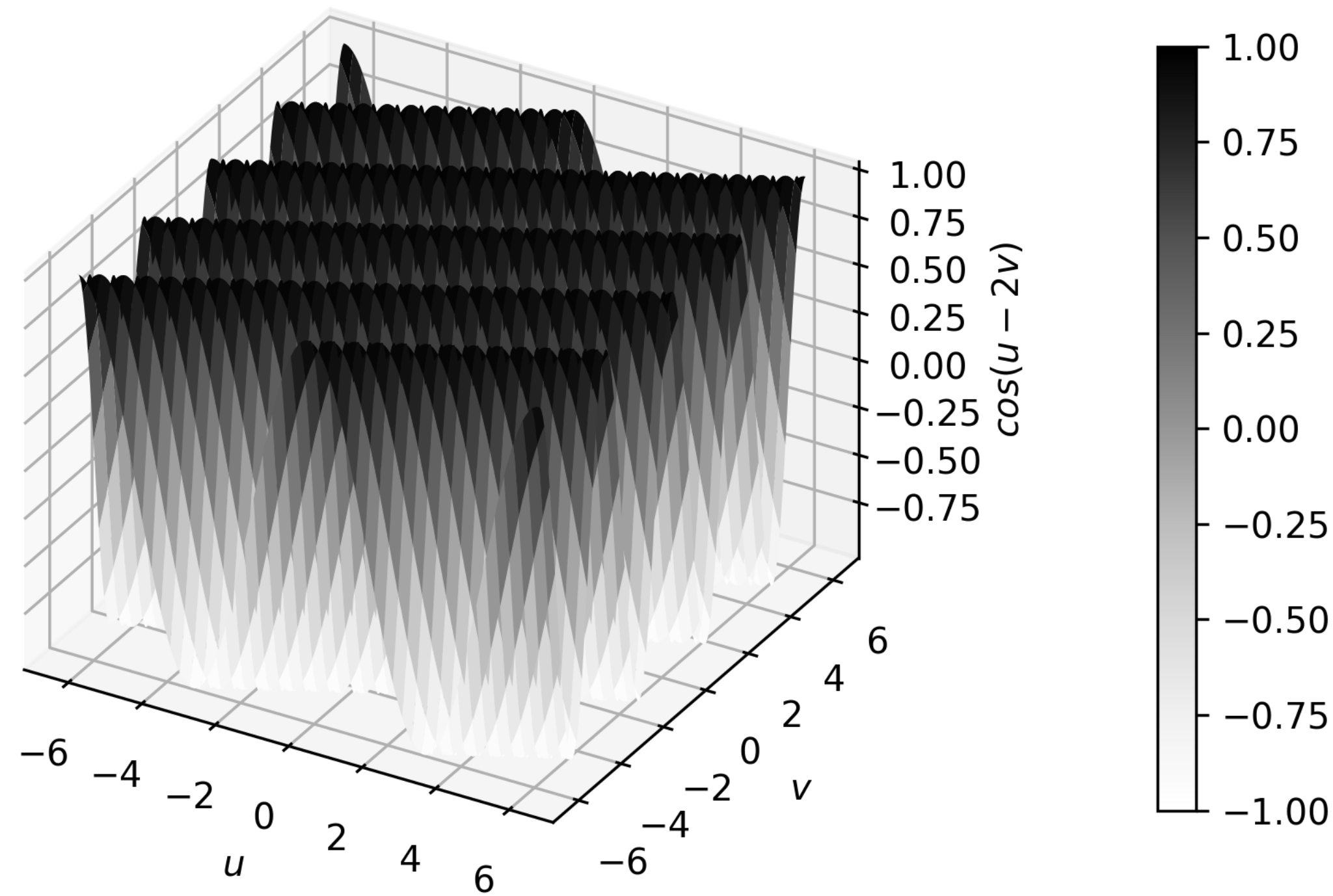


$\cos(u-v)$ is constant where $u-v$ is constant

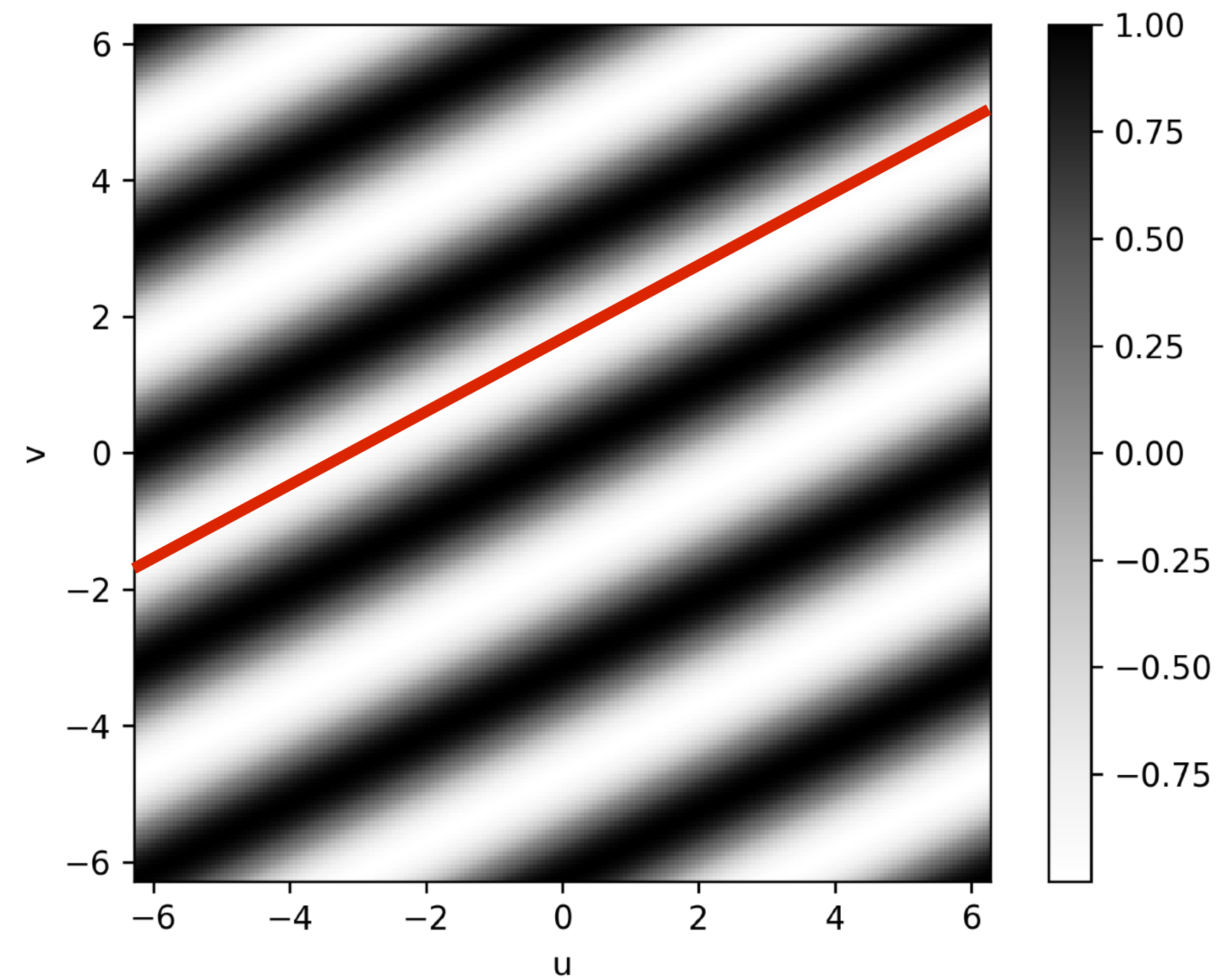


2D Sinusoidal Functions

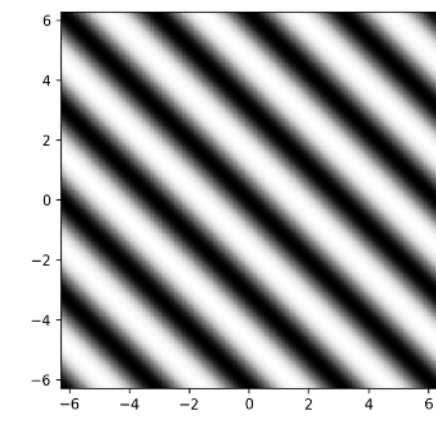
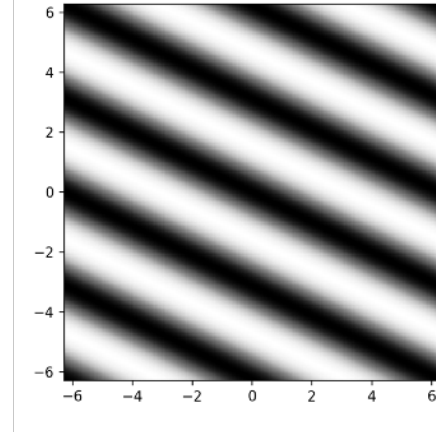
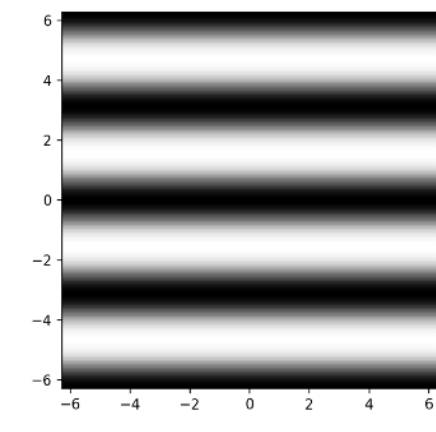
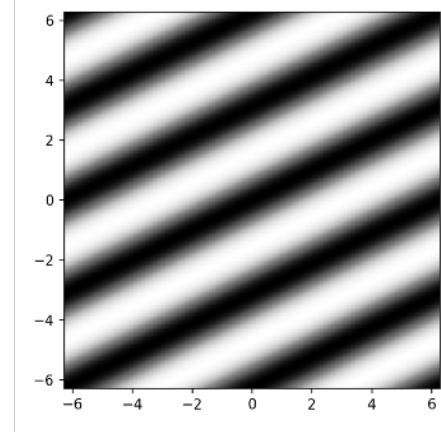
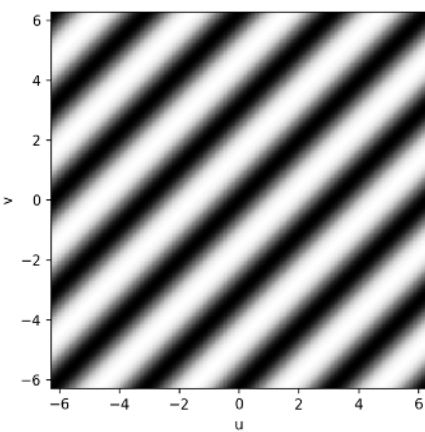
$\cos(u-2v)$



$\cos(u-2v)$ has the same value
with $u-2v$ is constant

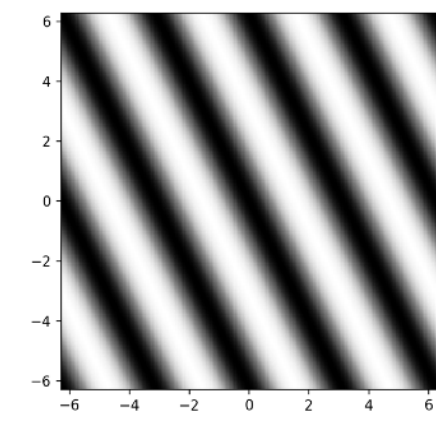
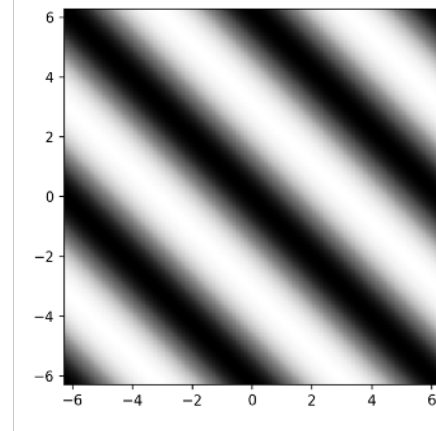
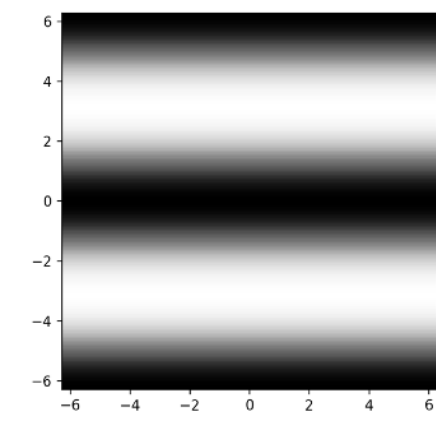
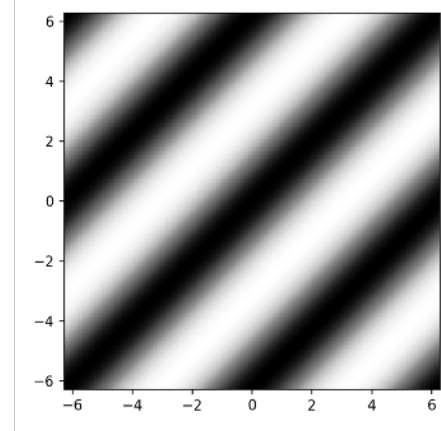
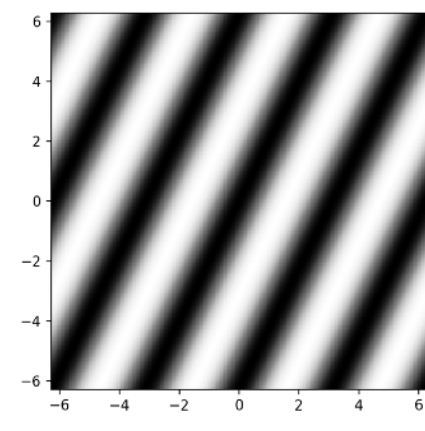


$$\cos(-2u+2v)$$



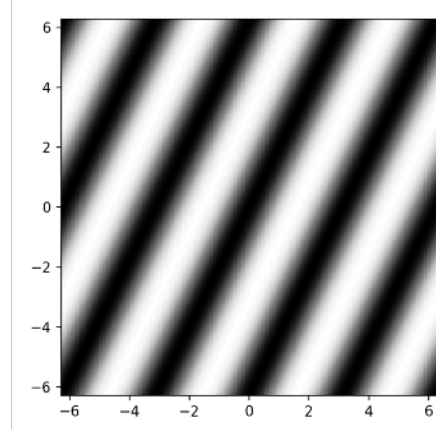
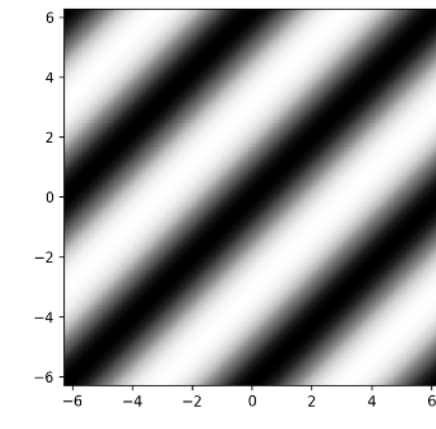
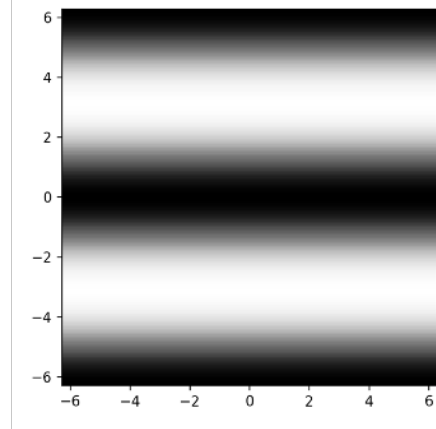
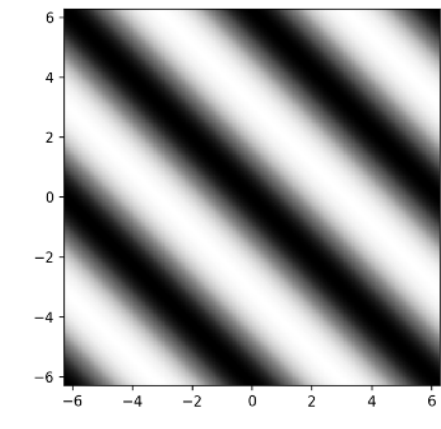
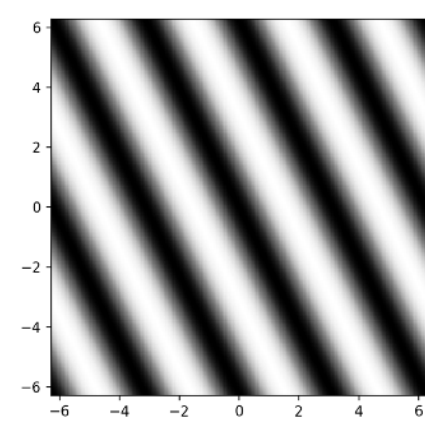
$$\cos(2u+2v)$$

$$\cos(-2u+0v)$$



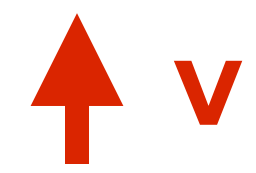
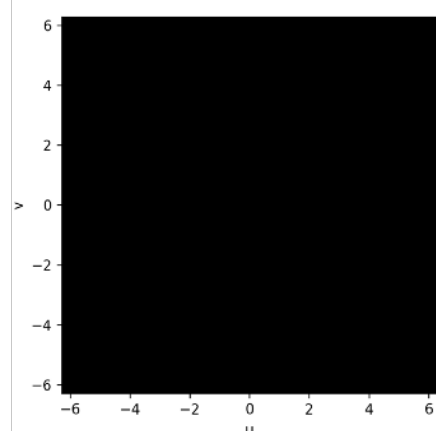
$$\rightarrow u$$

$$\cos(-2u-2v)$$



$$\cos(2u-2v)$$

$$\cos(0u-2v)$$



-

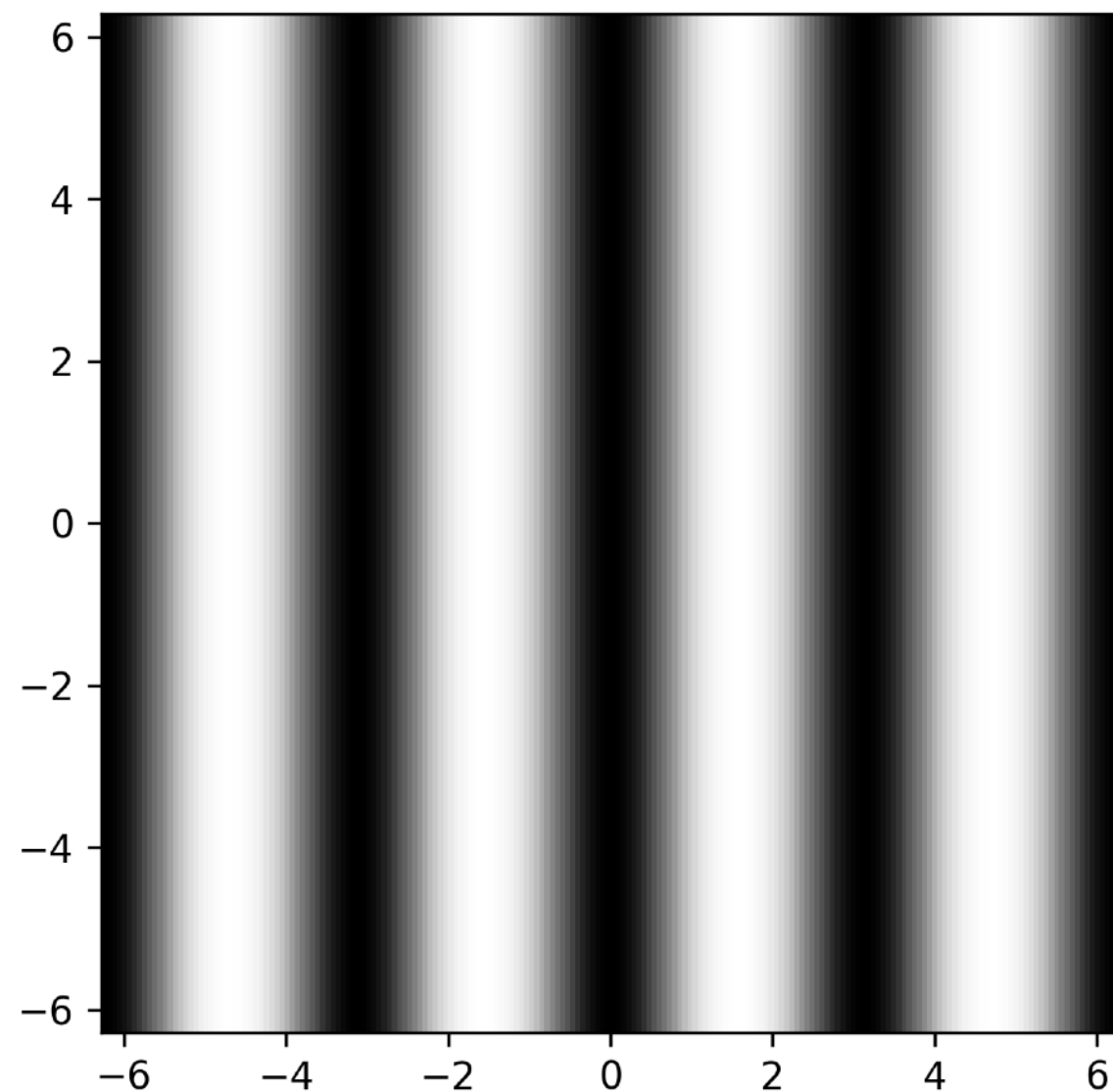
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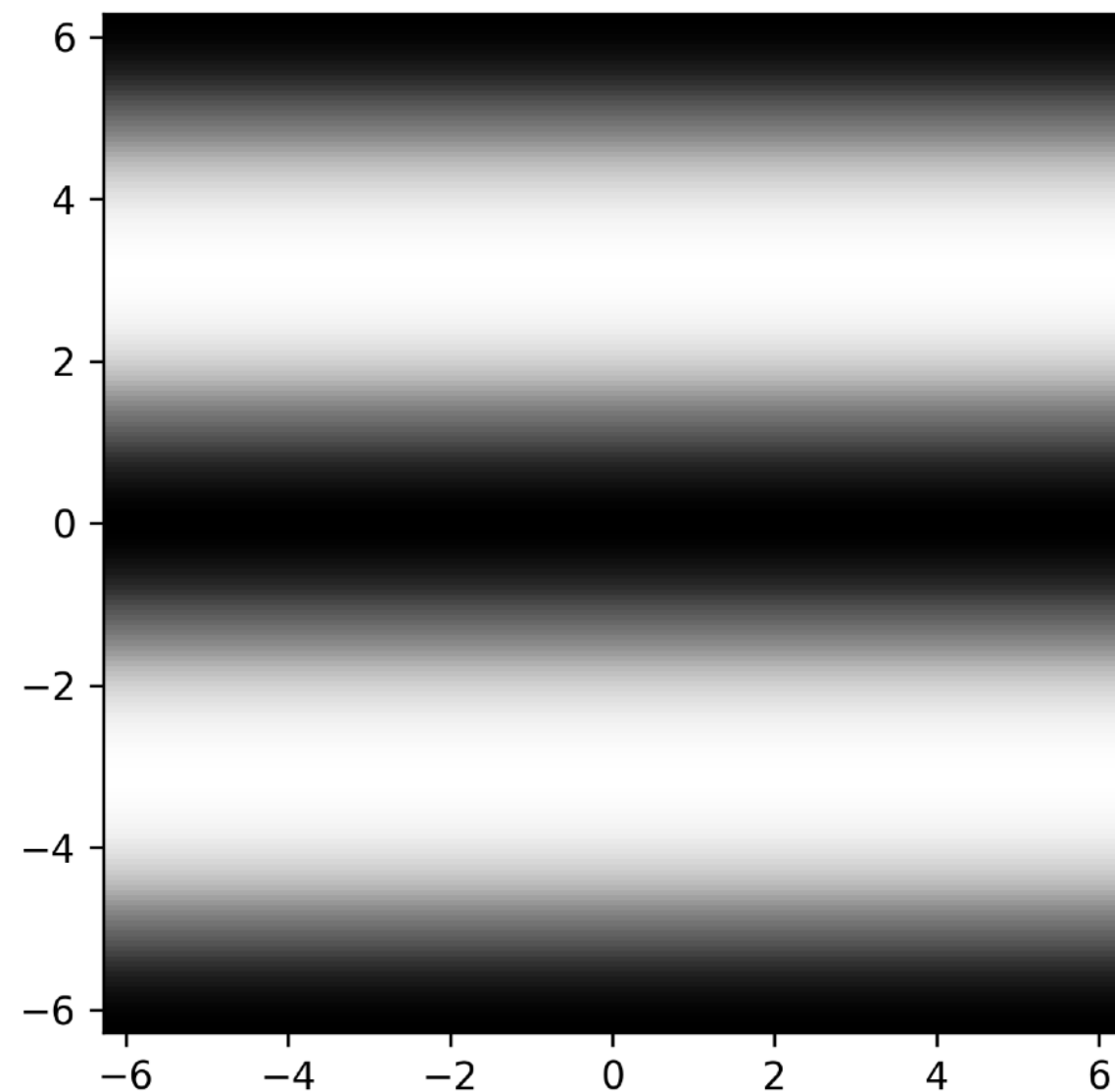
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2D Frequency Intuitions

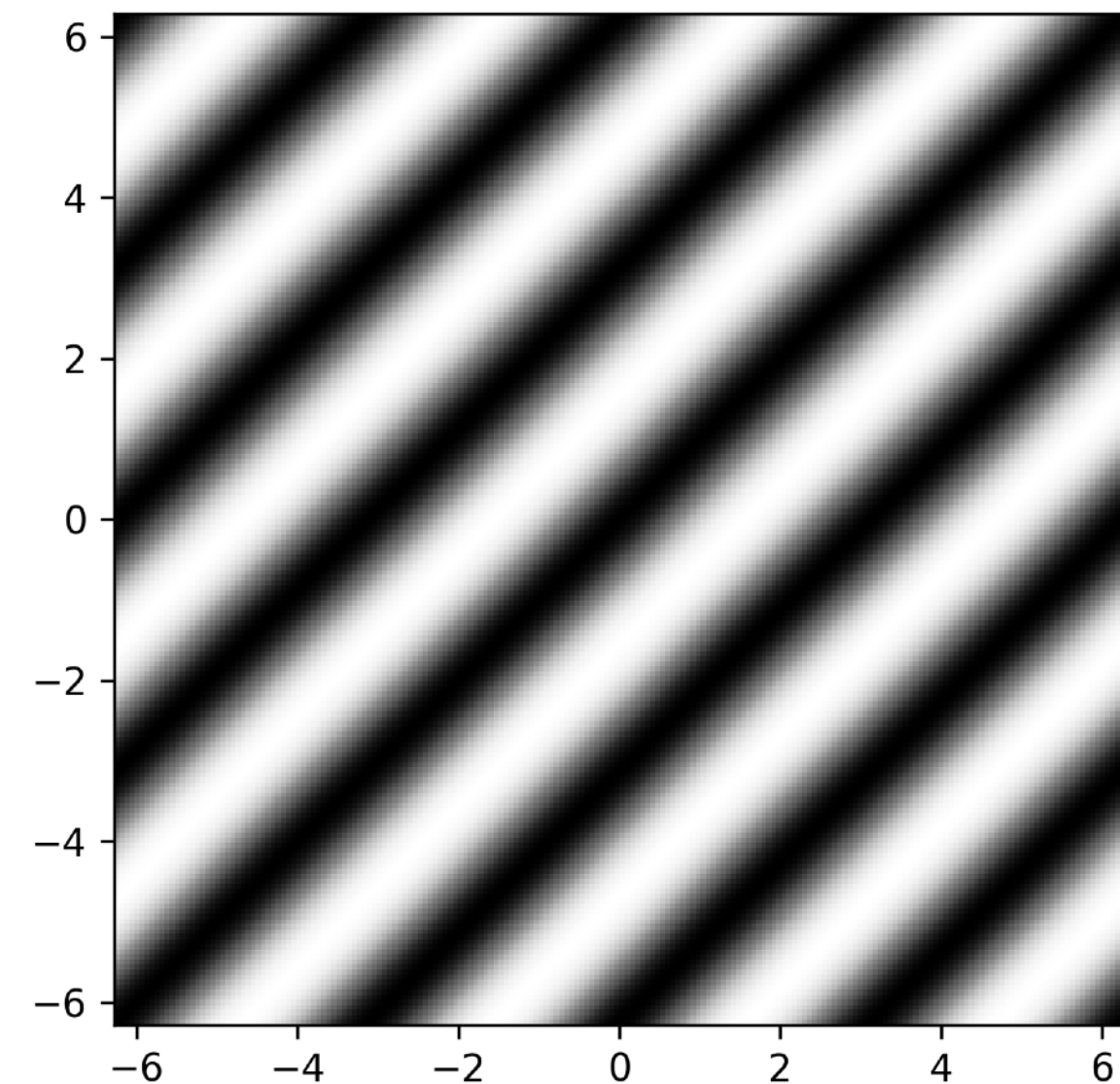
- 2D frequencies characterize the image spatial changes in horizontal and vertical directions
 - Smooth changes \rightarrow low frequencies
 - Sharp changes \rightarrow high frequencies



$\cos(2u+0v)$



$\cos(0u-v)$



$\cos(2u-2v)$

Image = 2D Signal

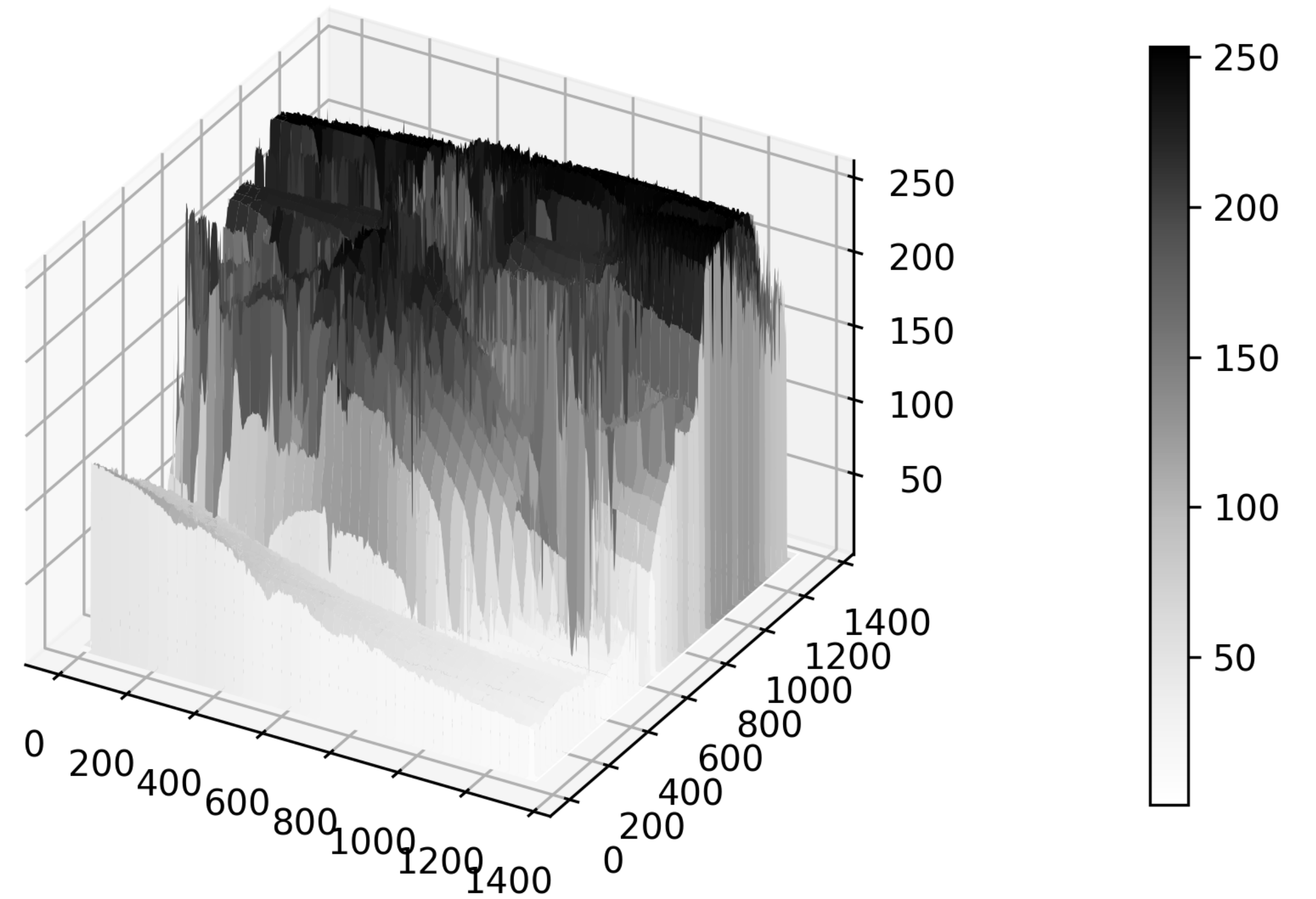


Image = 2D Signal



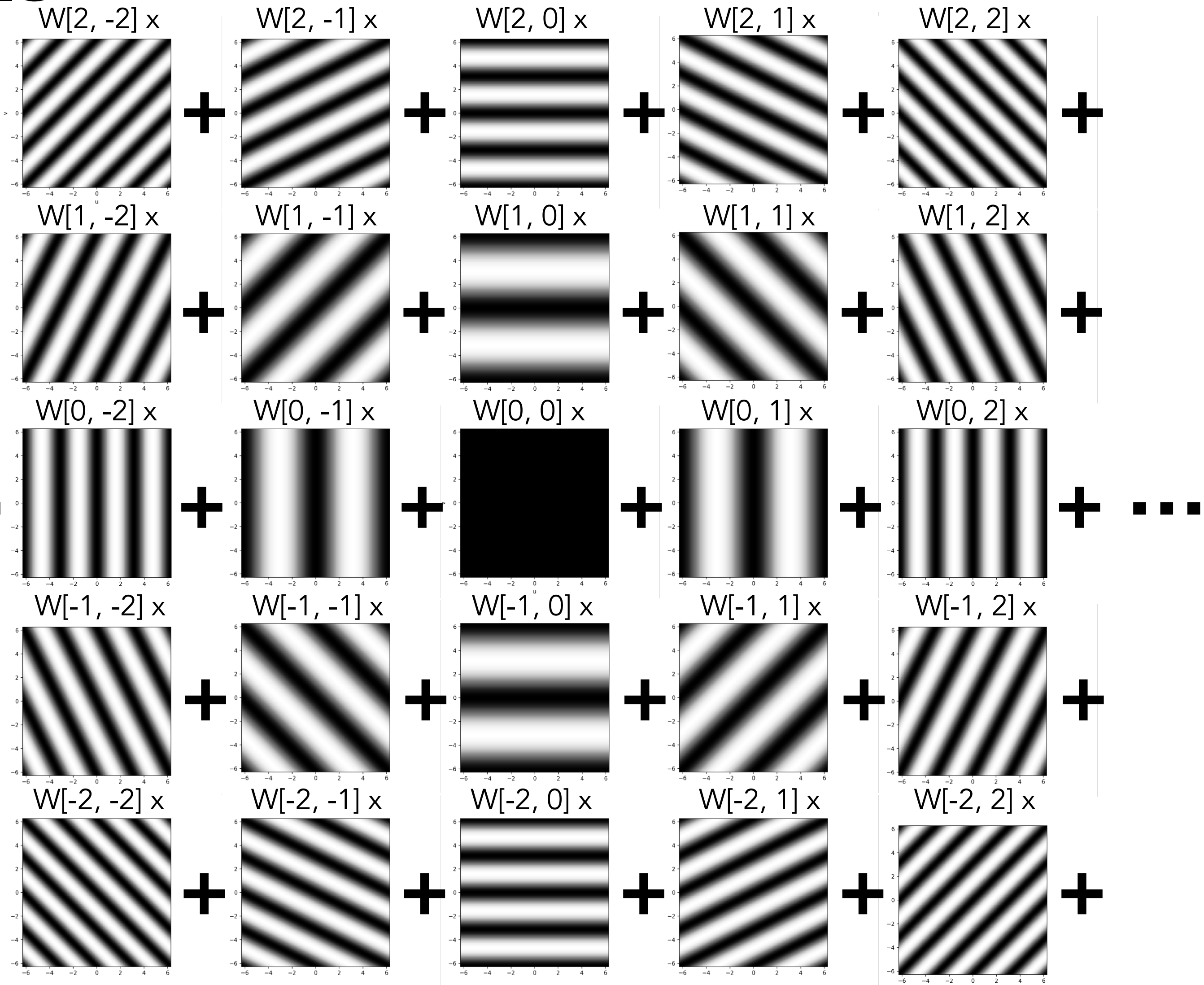
- An image is nothing more than a (complicated) 2D function $f(x, y)$
- $f(x, y)$ maps a pixel coordinates $[x, y]$ to its pixel value
- Can a 2D signal be expressed as a weighted sum of a set of 2D sinusoidal functions?

Decomposing 2D Signals



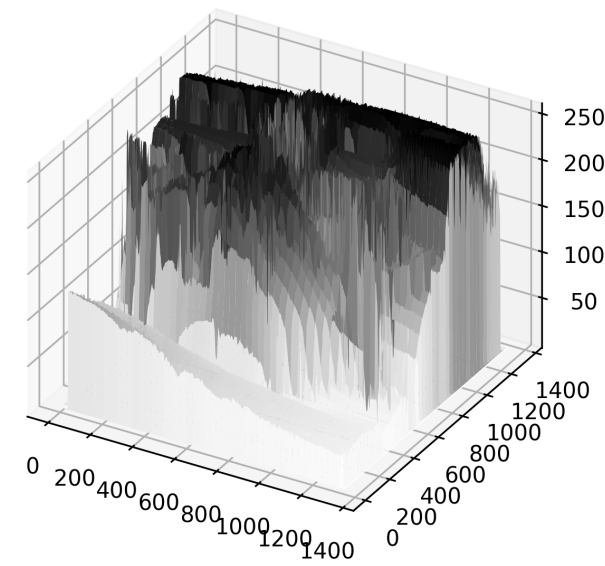
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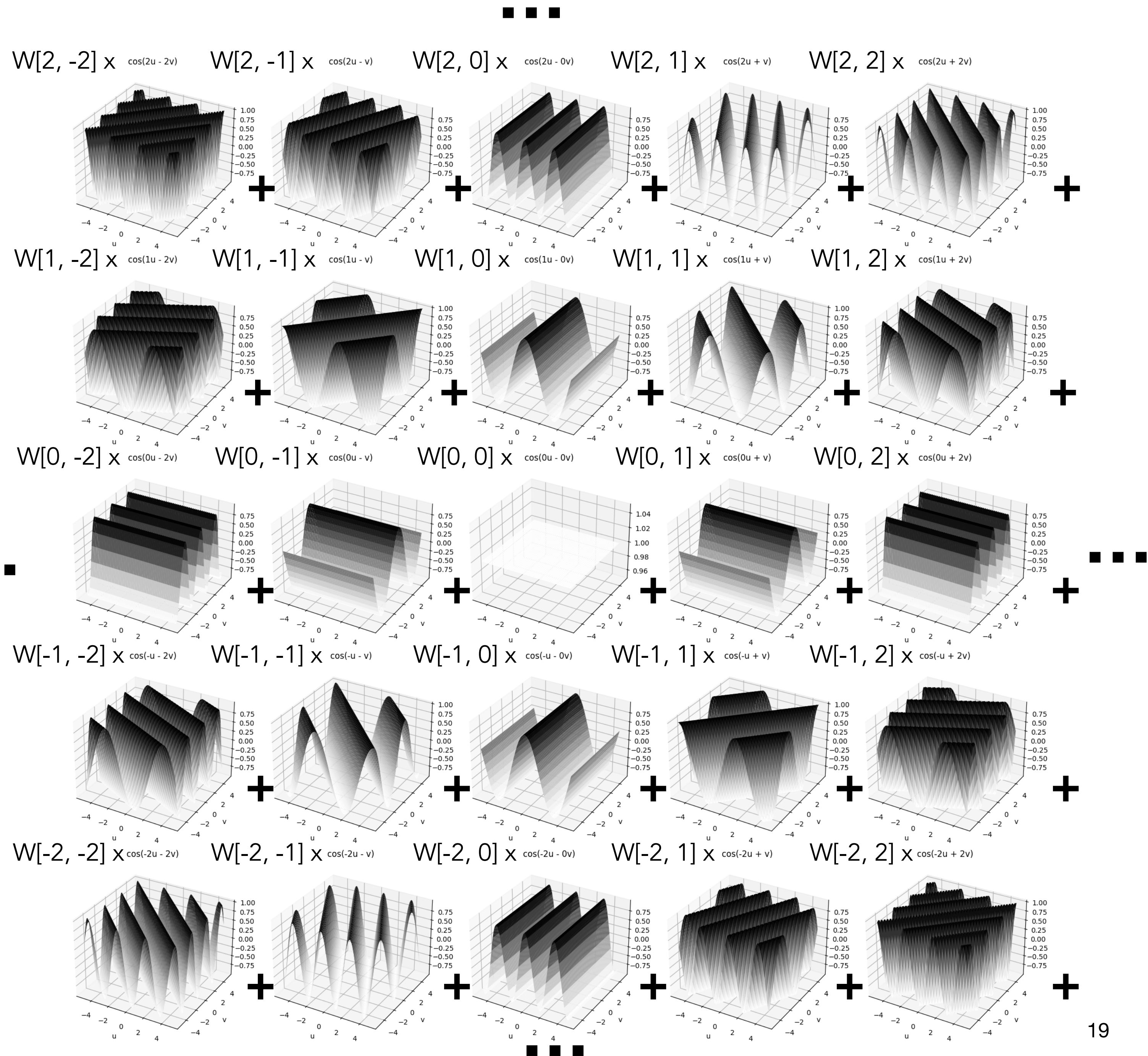


...

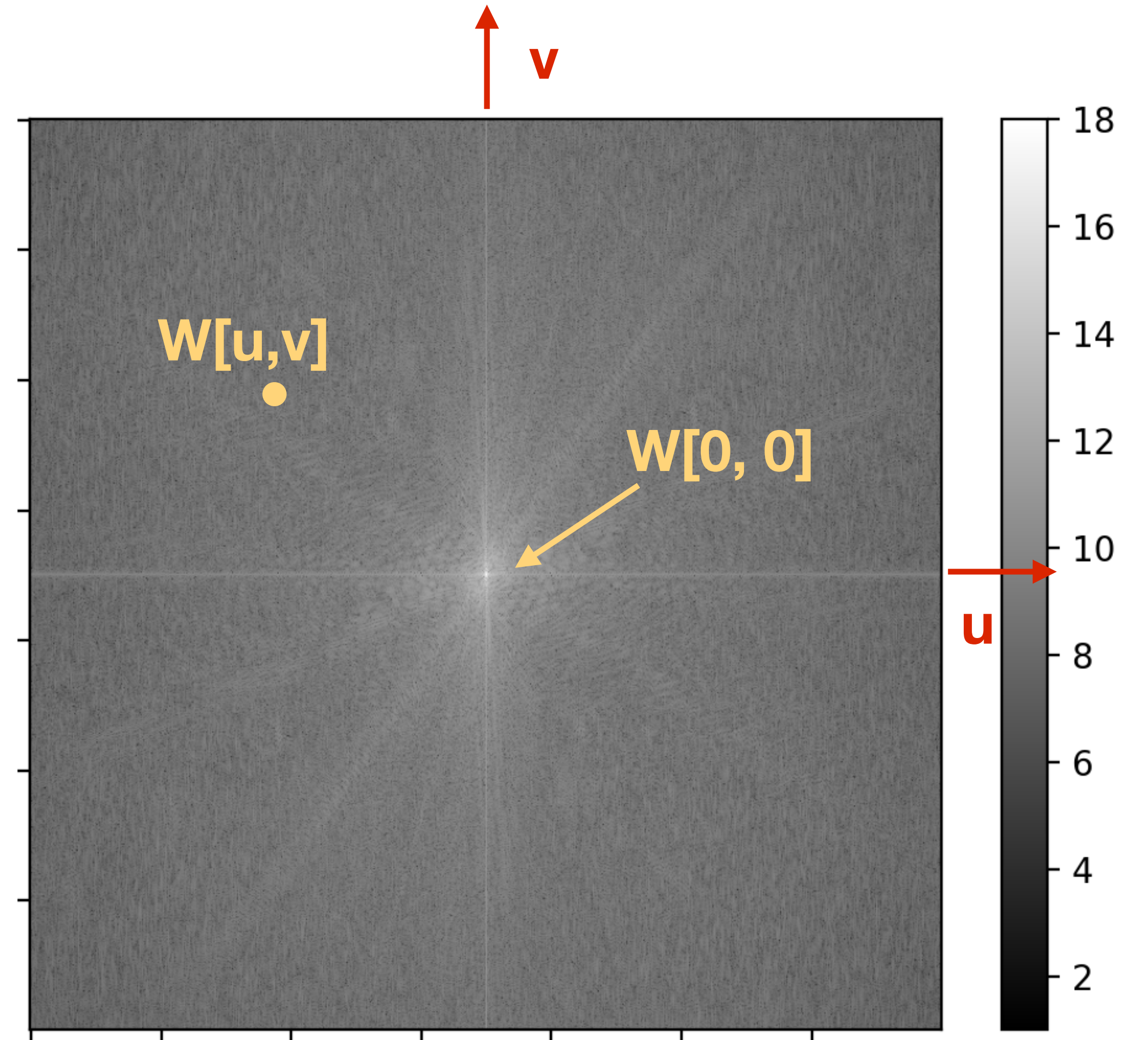
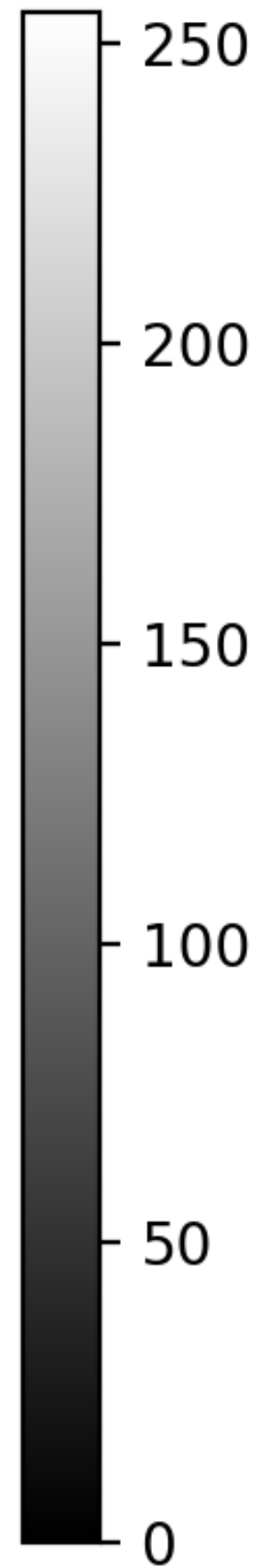
Another Perspective



= ...



Plotting the 2D Weights*



* the weights are actually complex values, so we plot their magnitudes here.

Reconstruction By Adding Individual Terms

Original Image



Reconstruction with Terms
within $\cos[-1,1]$ and $\cos[1,-1]$



Reconstruction By Adding Individual Terms

Original Image



Reconstruction with Terms
within $\cos[-3,3]$ and $\cos[3,-3]$



Reconstruction By Adding Individual Terms

Original Image



Reconstruction with Terms
within $\cos[-5,5]$ and $\cos[5,-5]$



Reconstruction By Adding Individual Terms

Original Image



Reconstruction with Terms
within $\cos[-7,7]$ and $\cos[7,-7]$



Reconstruction By Adding Individual Terms

Original Image



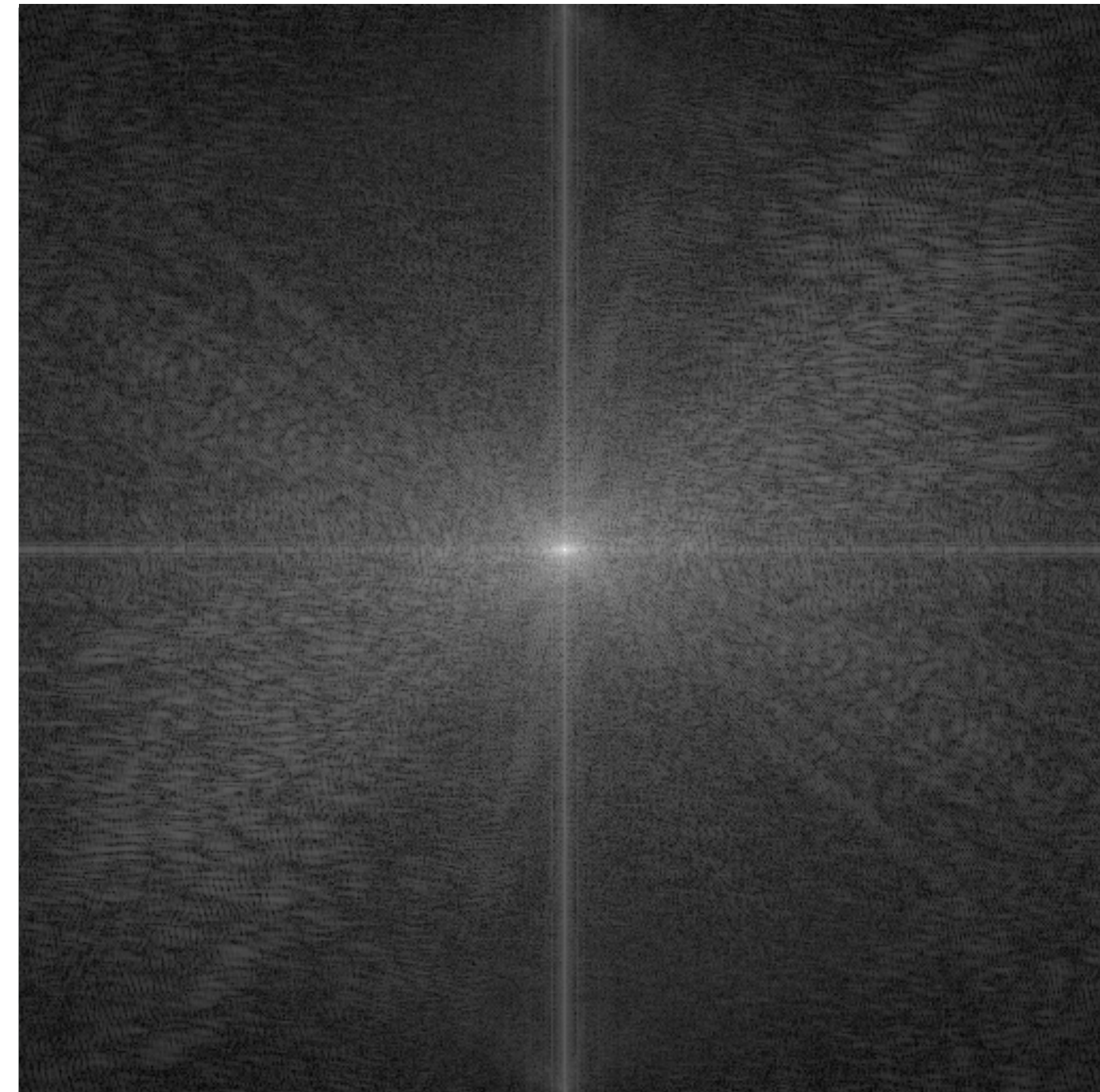
Reconstruction with Terms
within $\cos[-50,50]$ and $\cos[50,-50]$



Manipulating the Weights



Spatial Domain

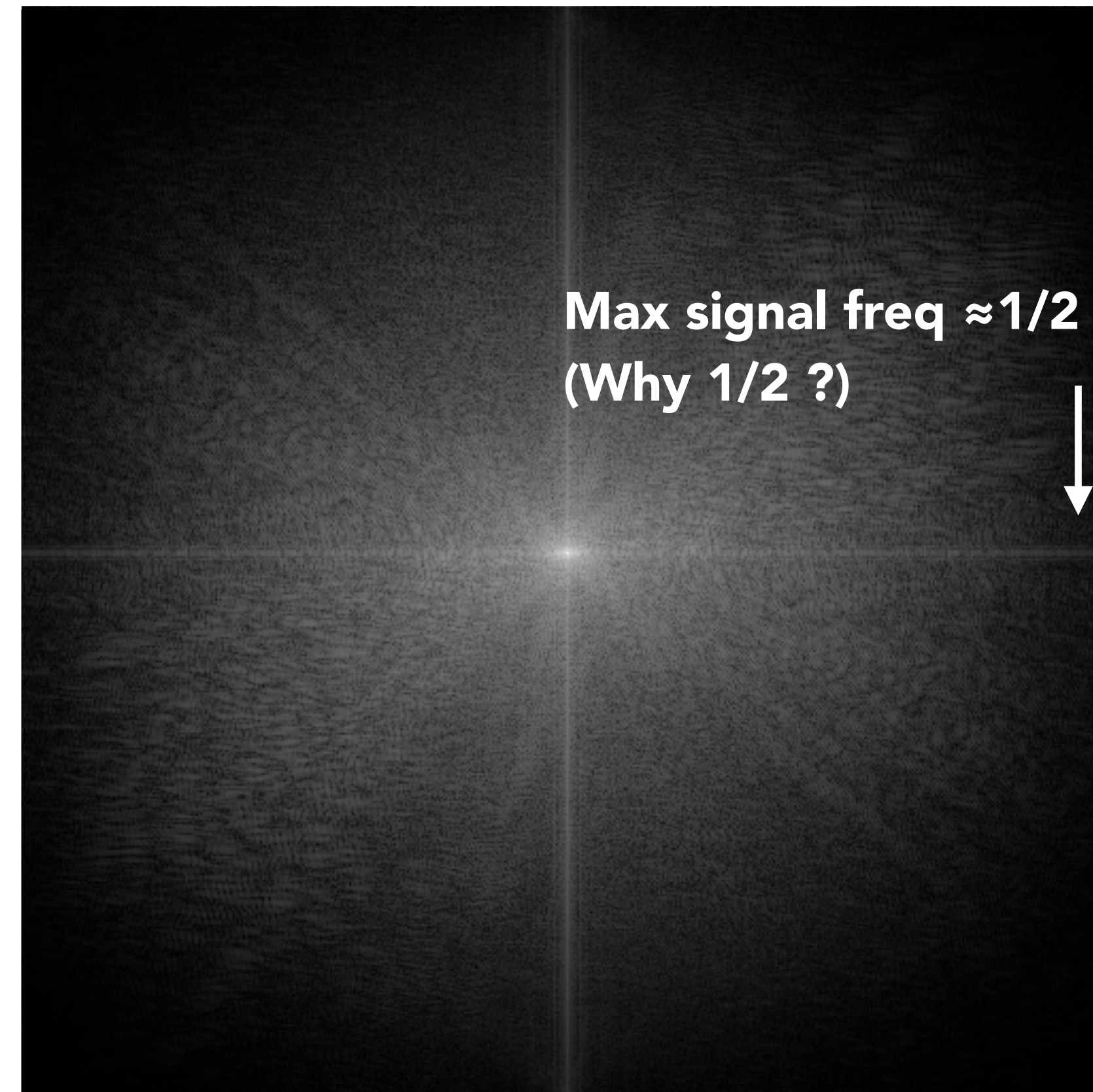


Frequency Domain

Manipulating the Weights



Spatial Domain

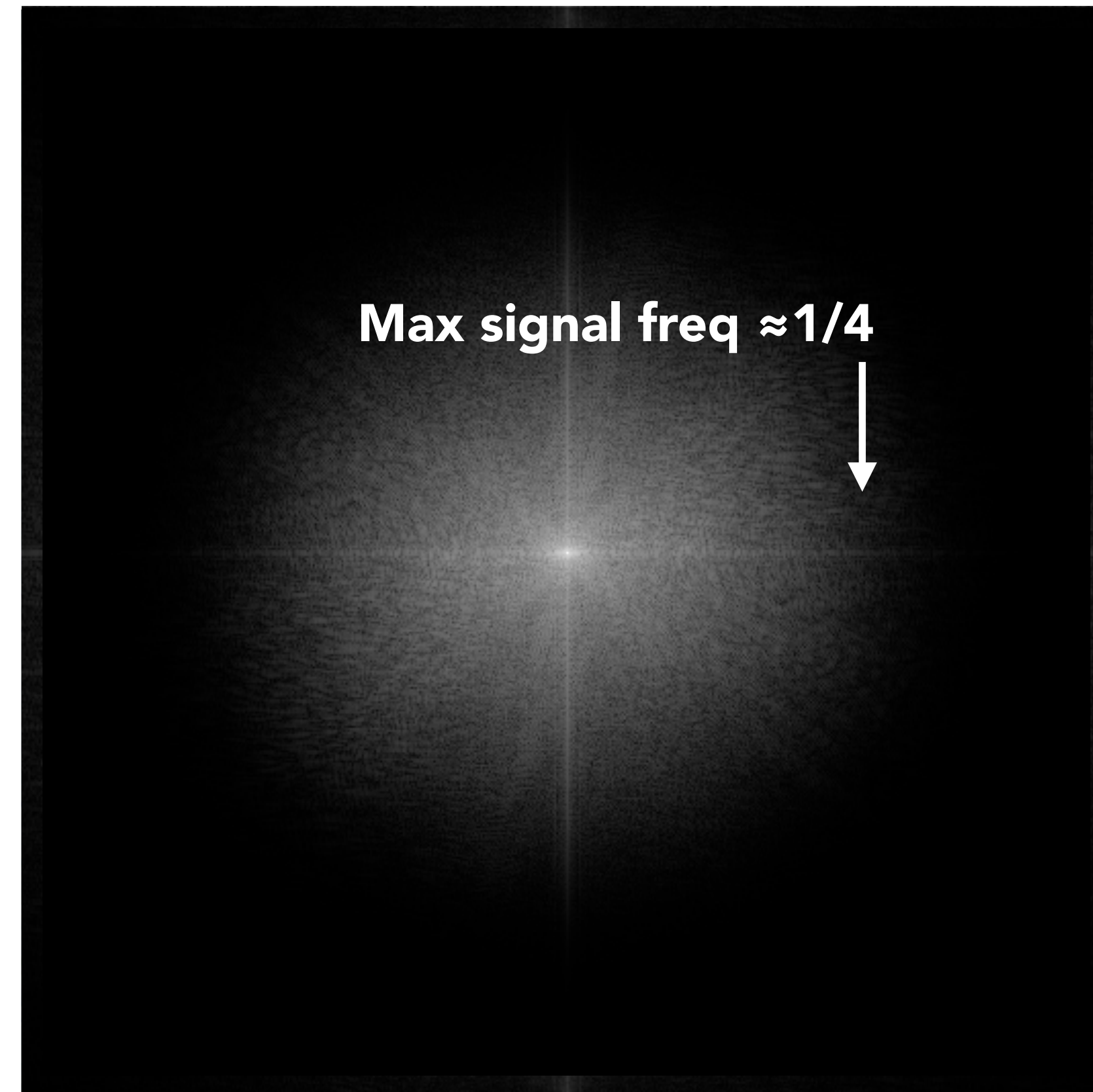


Frequency Domain

Manipulating the Weights



Spatial Domain

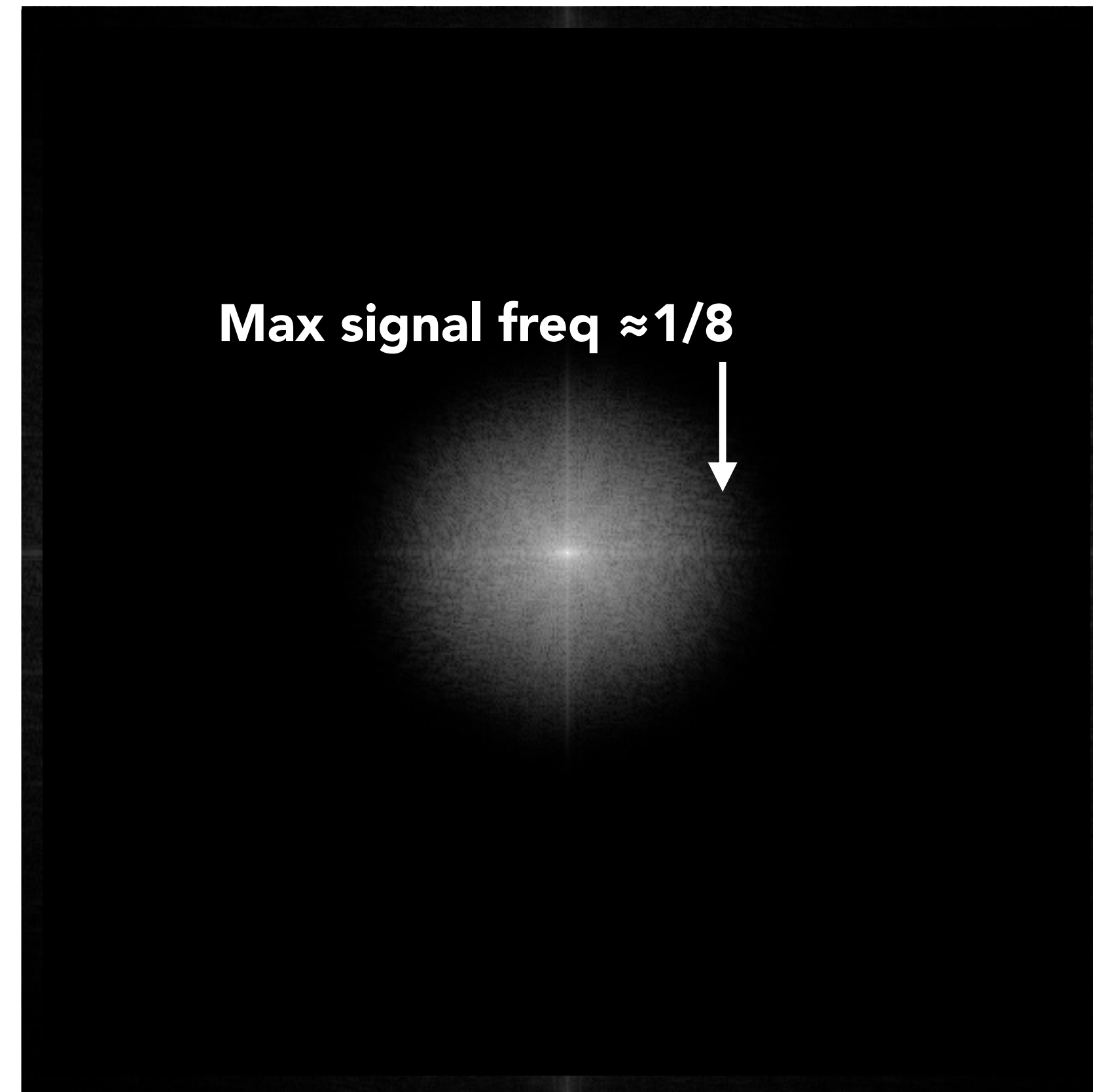


Frequency Domain

Manipulating the Weights



Spatial Domain

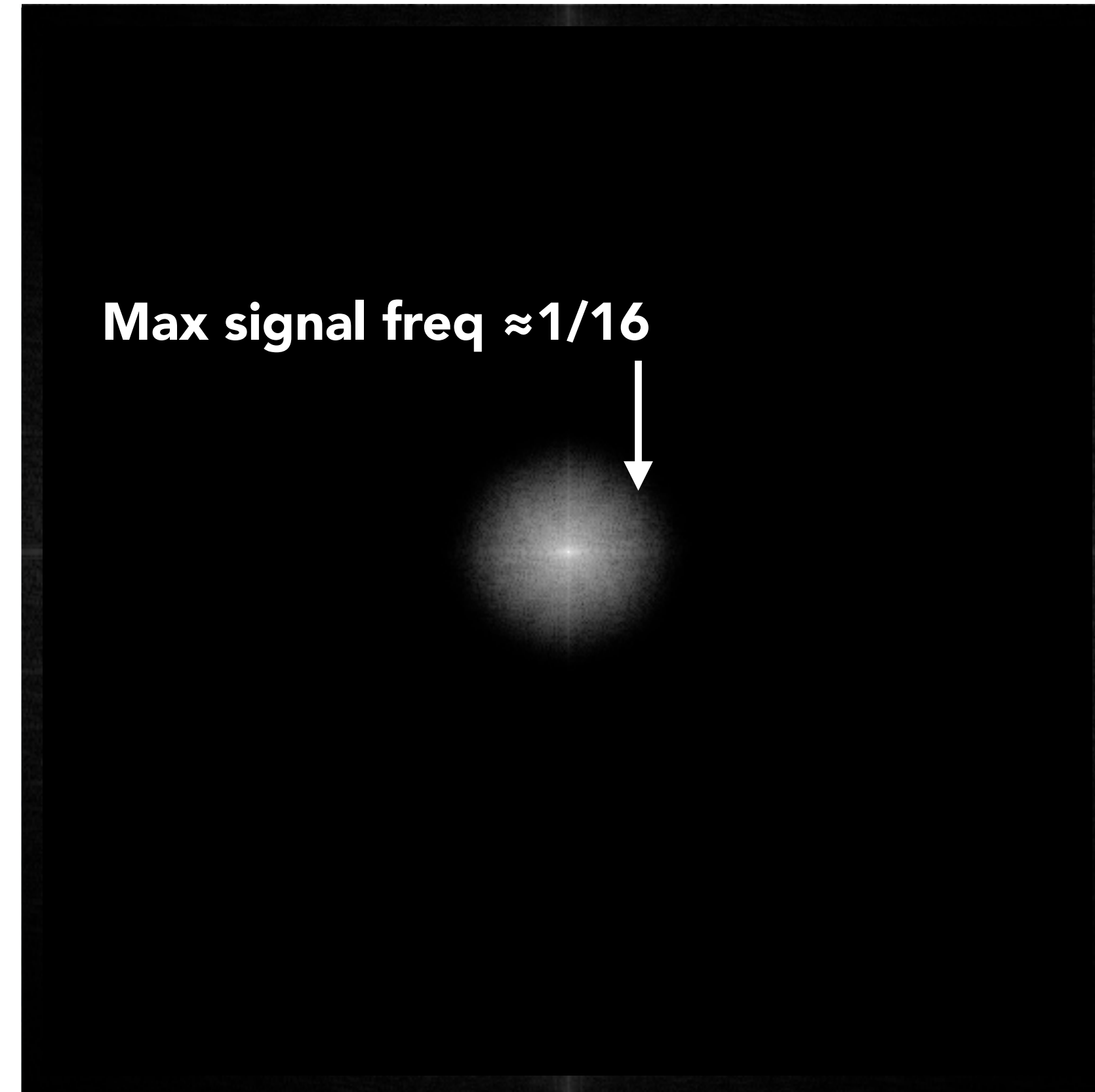


Frequency Domain

Manipulating the Weights



Spatial Domain

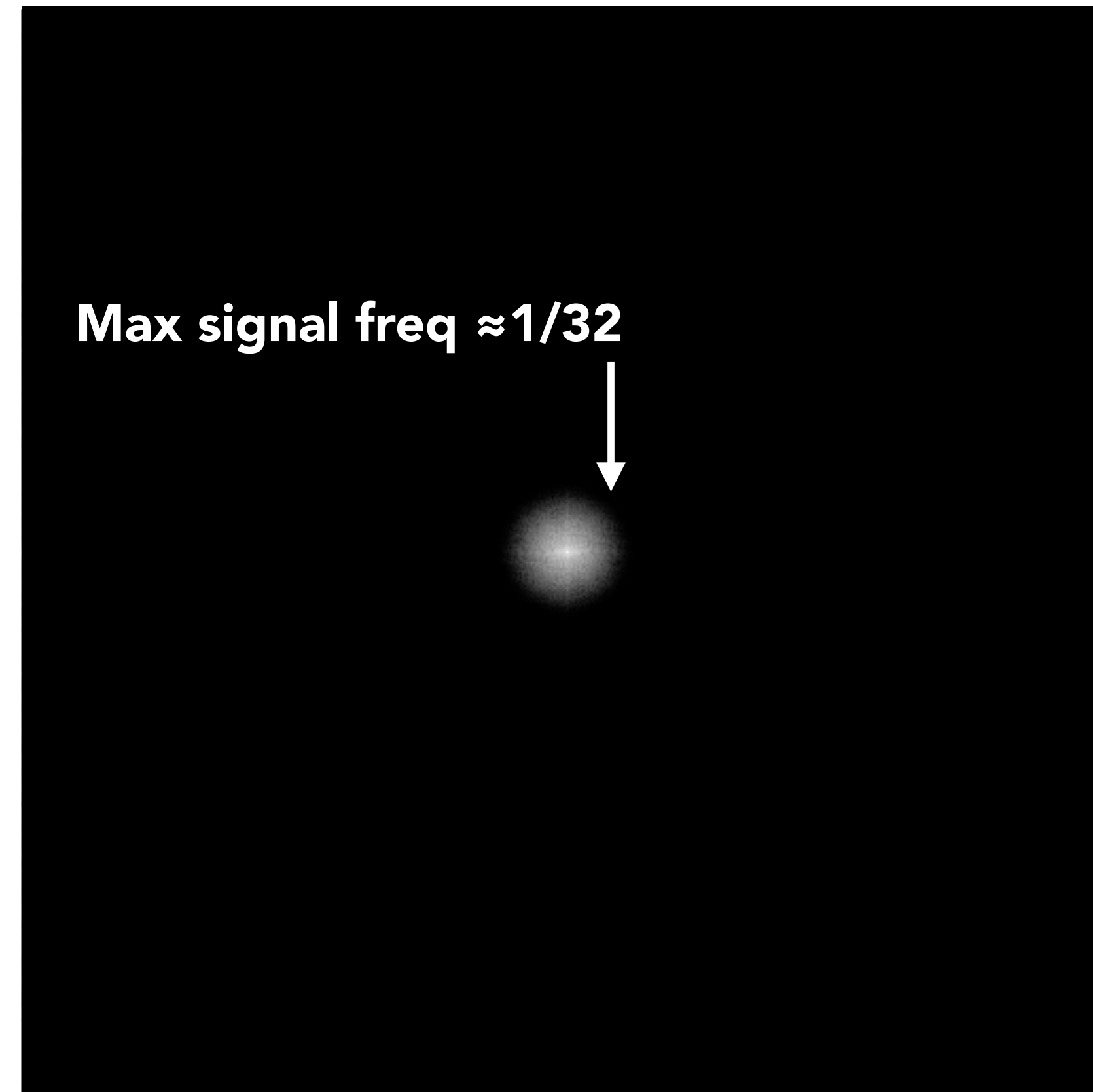


Frequency Domain

Manipulating the Weights



Spatial Domain

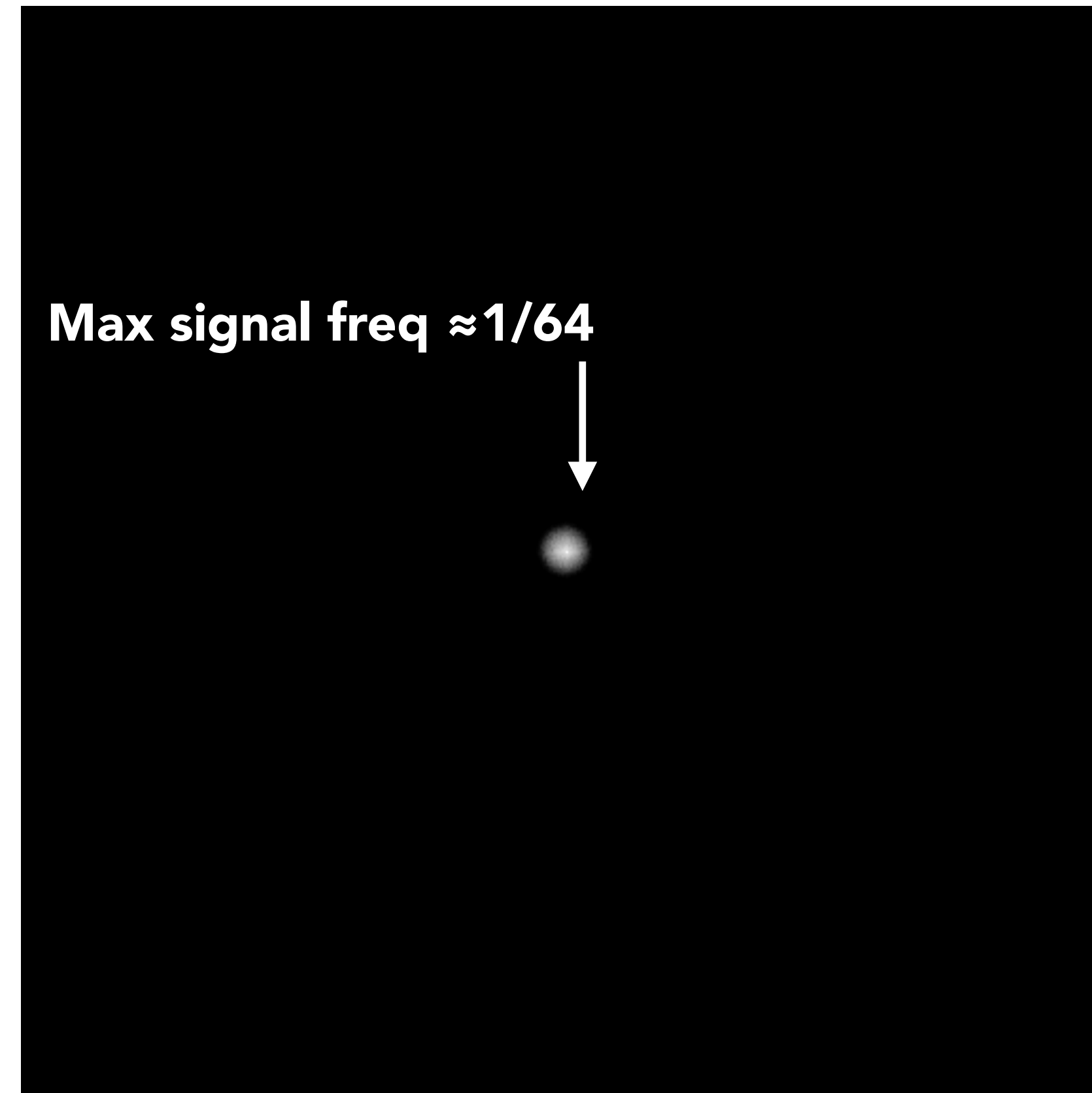


Frequency Domain

Manipulating the Weights

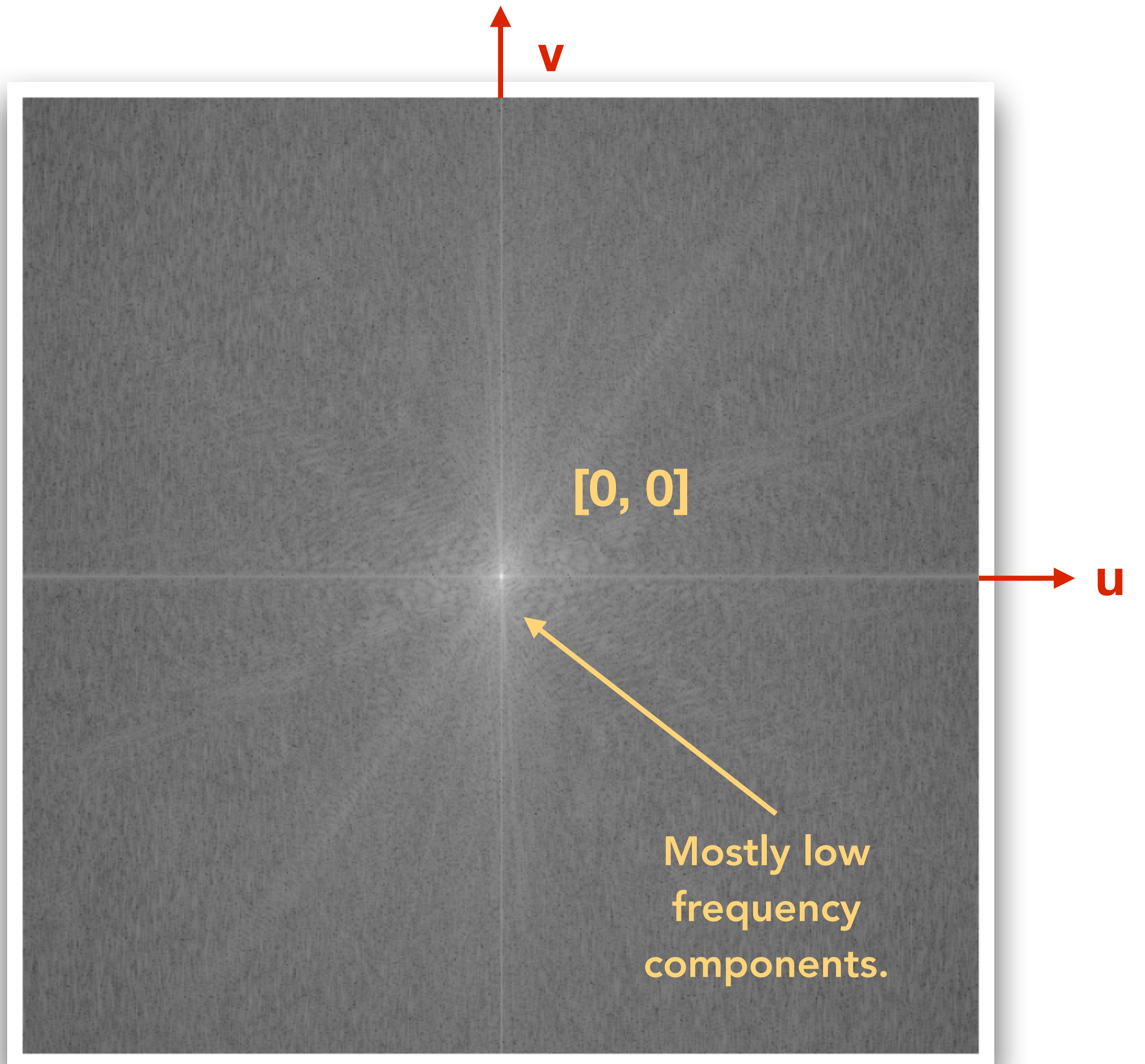


Spatial Domain

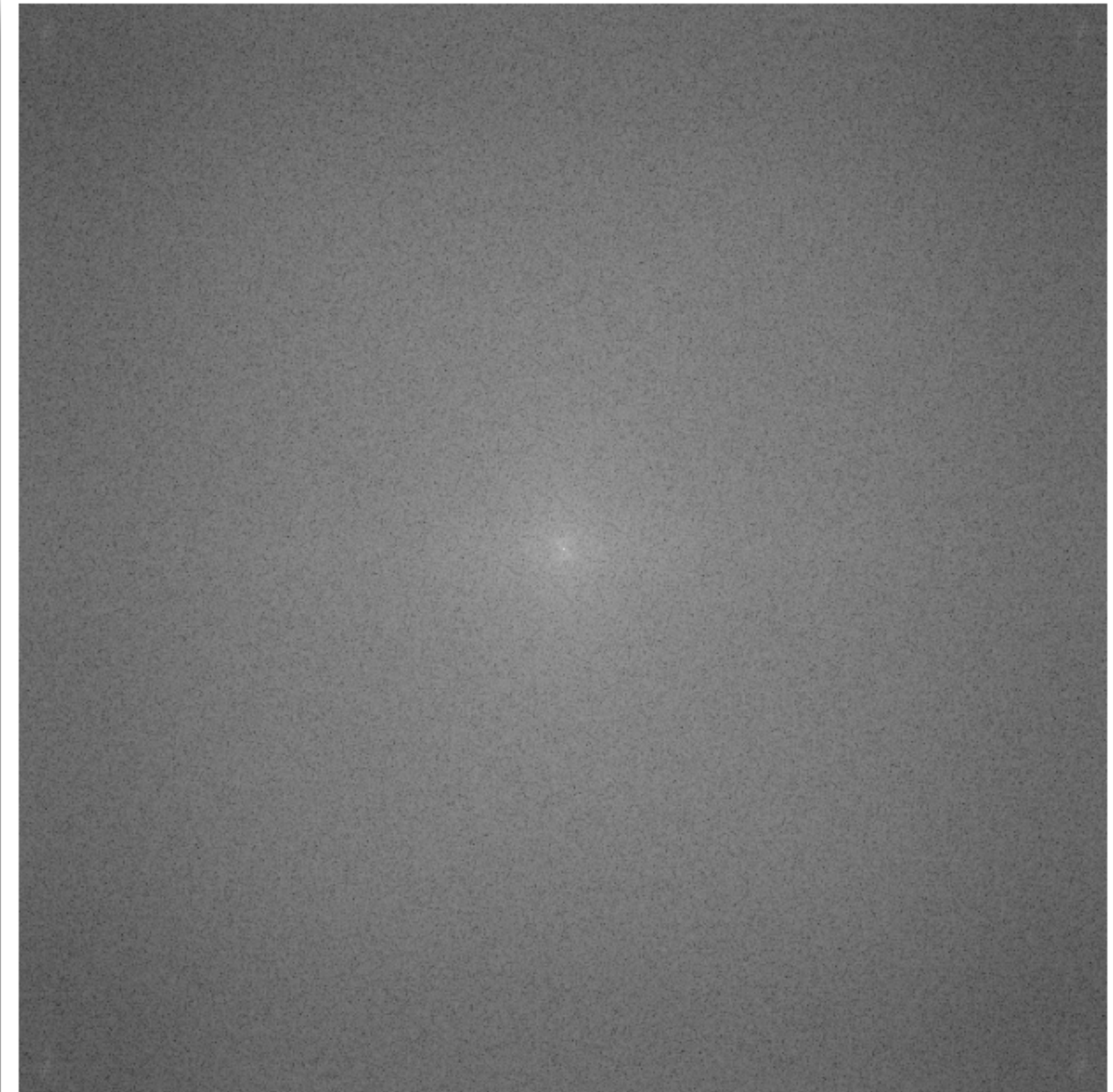
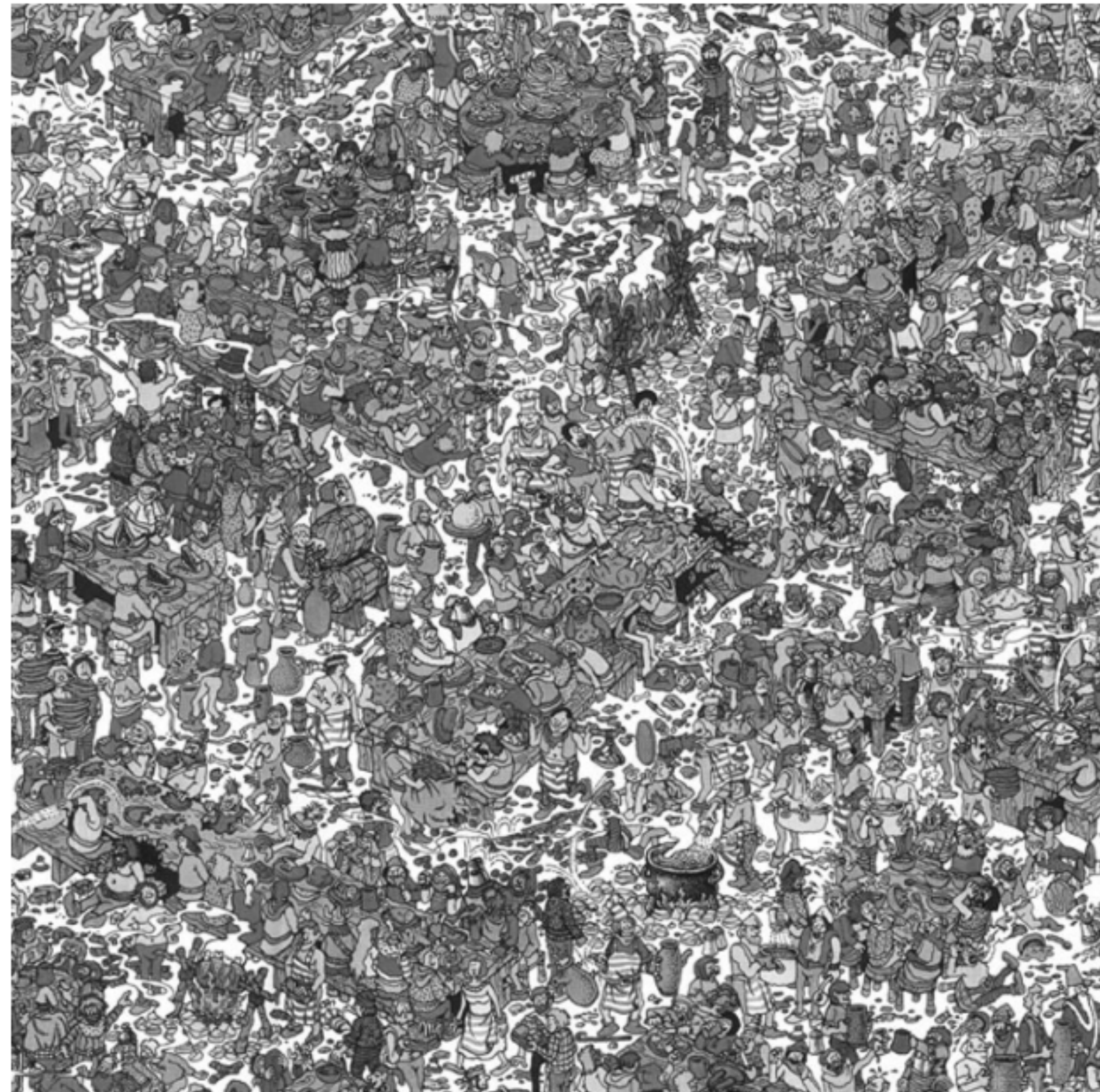


Frequency Domain

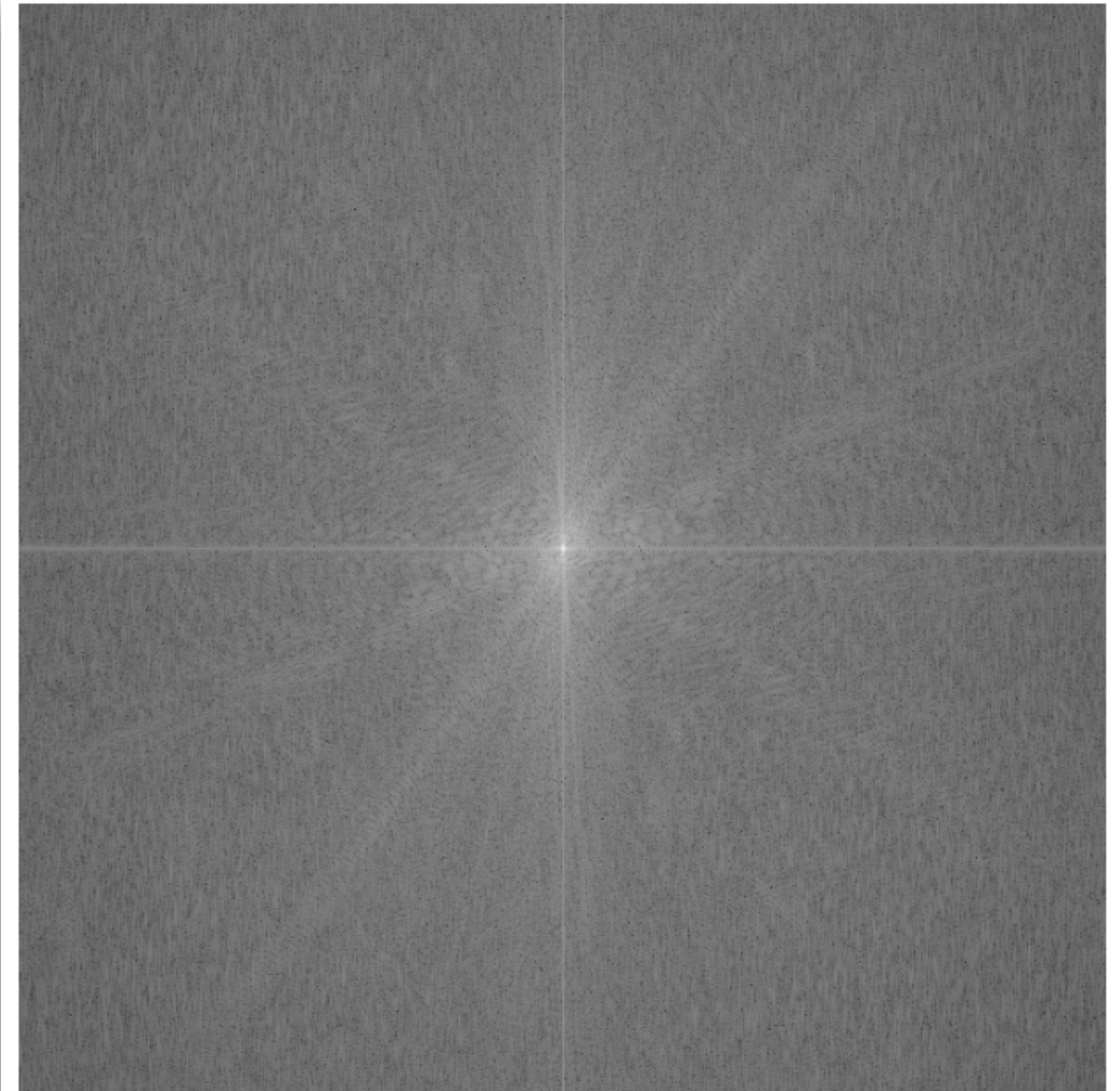
A Typical Photo



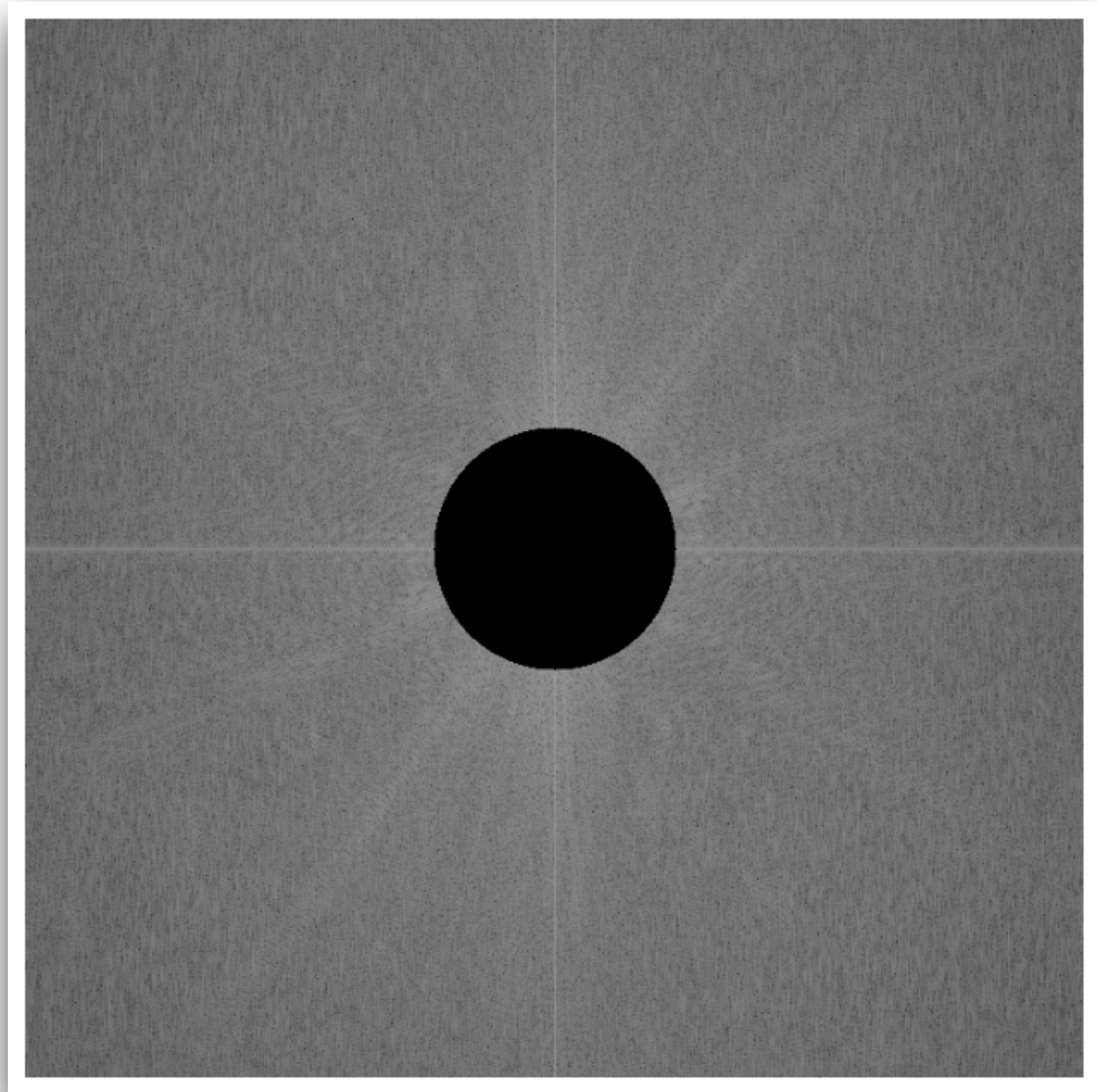
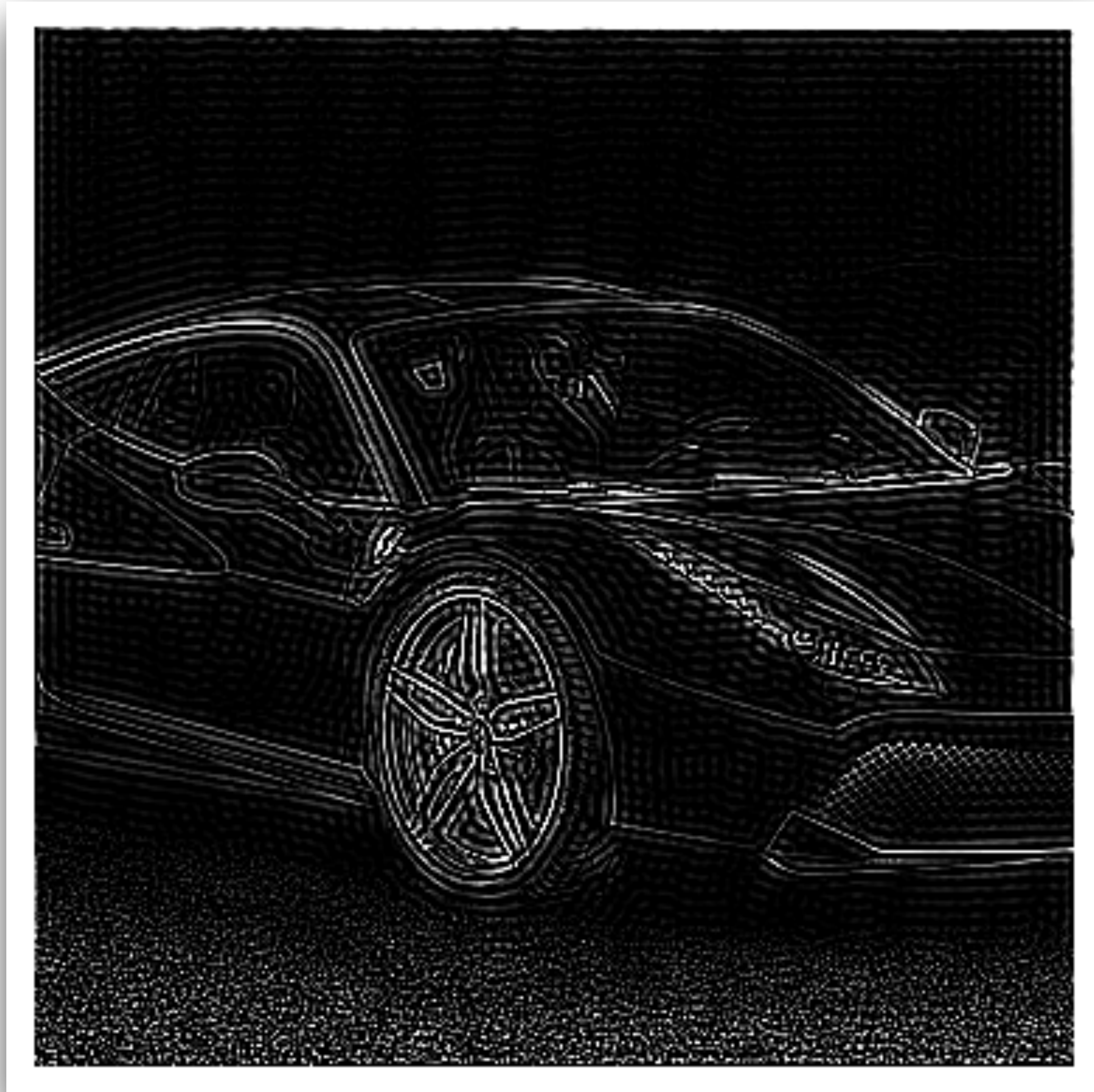
A Photo With Many High-Frequency Components



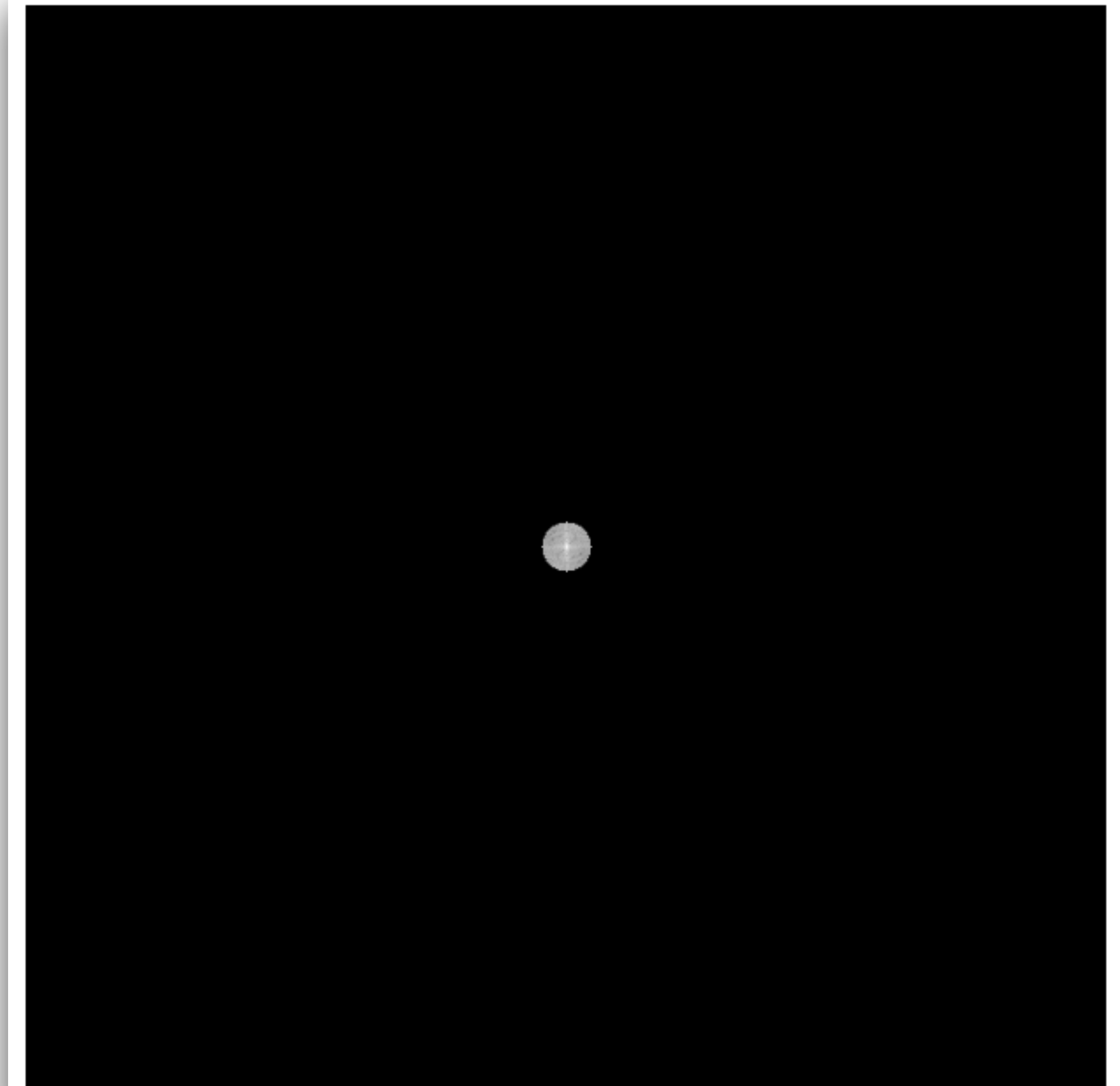
Original Photo



High-Pass Filter (Removing Low-Freq Components)

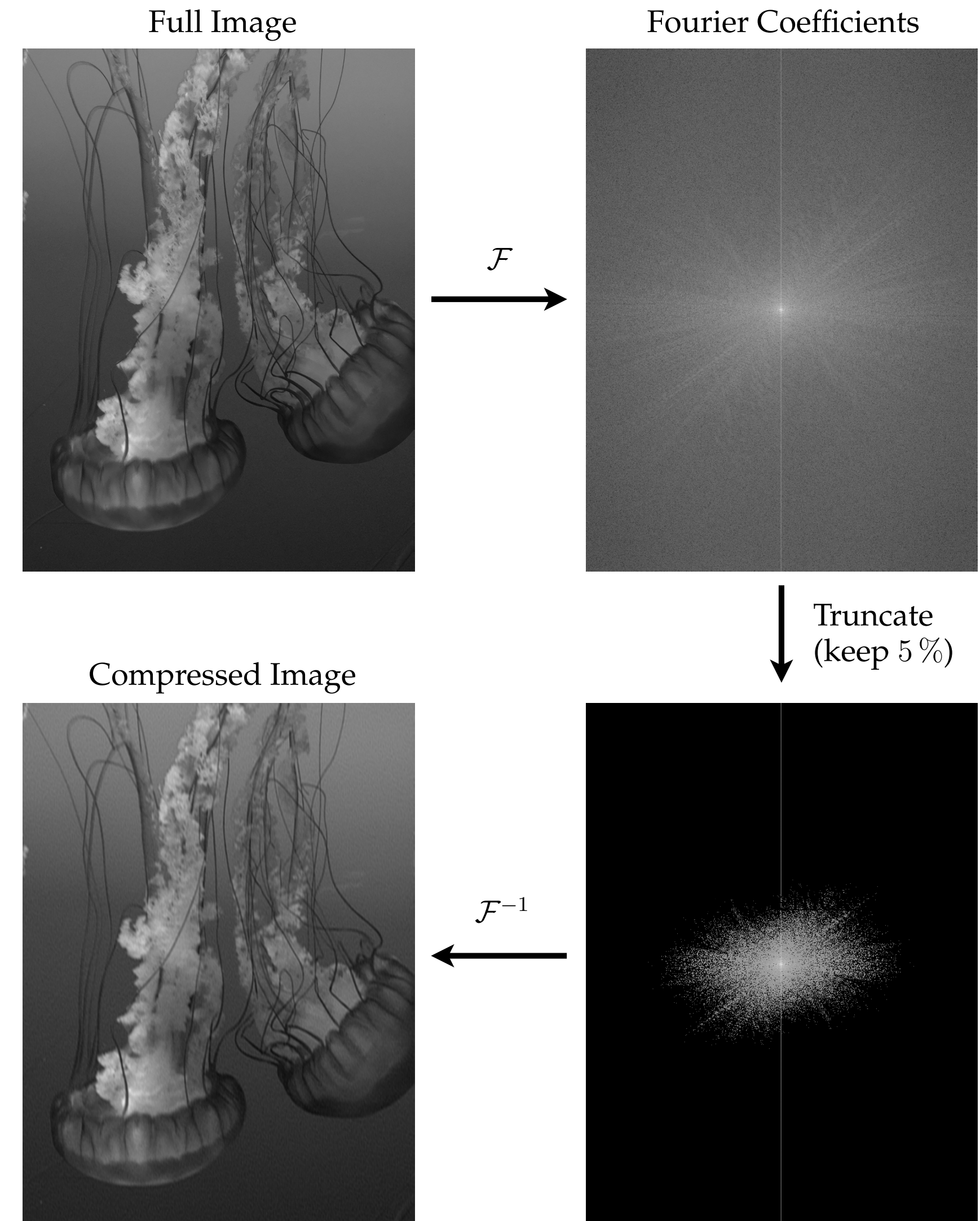


Low-Pass Filter (Removing High-Freq Components)



An Application: Image Compression

- Steps:
 - Calculate the “weight” or coefficient matrix.
 - Zero-out small coefficients
 - Then reconstruct the image using the remaining coefficients
- JPEG uses something very similar.



An Application: Image Compression

- Steps:
 - Calculate the “weight” or coefficient matrix.
 - Zero-out small coefficients
 - Then reconstruct the image using the remaining coefficients
- **JPEG uses something very similar.**

Full image



5.0% of FFT



1.0% of FFT

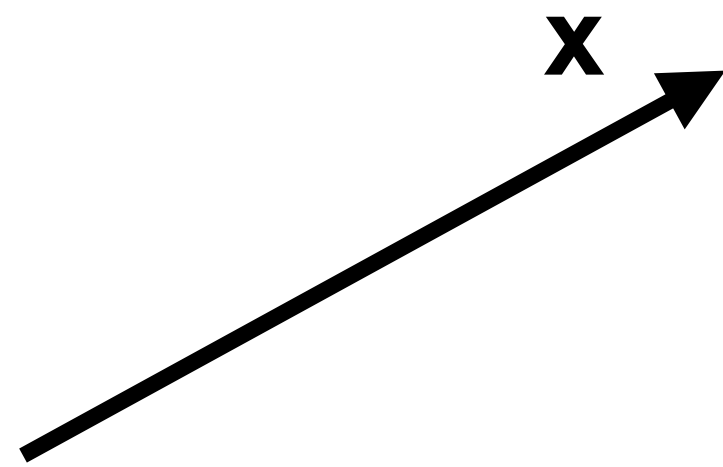


0.2% of FFT



How Does It Work?

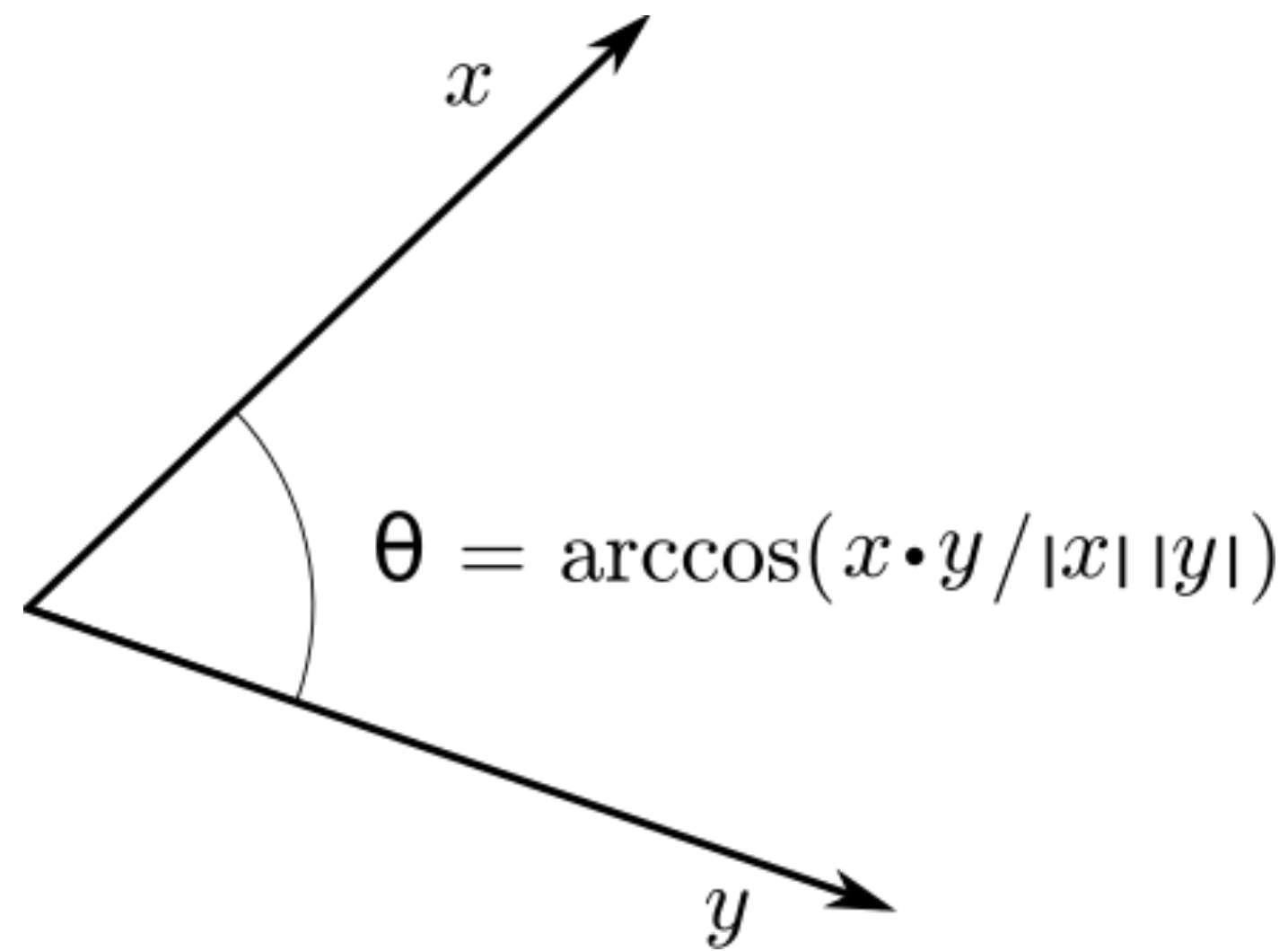
Vector Norm



Vector norm $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$

Unit vector $\hat{\mathbf{x}} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$

Inner/Dot Product



$$\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2]$$

$$\mathbf{y} = [\mathbf{y}_1, \mathbf{y}_2]$$

Geometric
definition

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta$$

Algebraic
definition

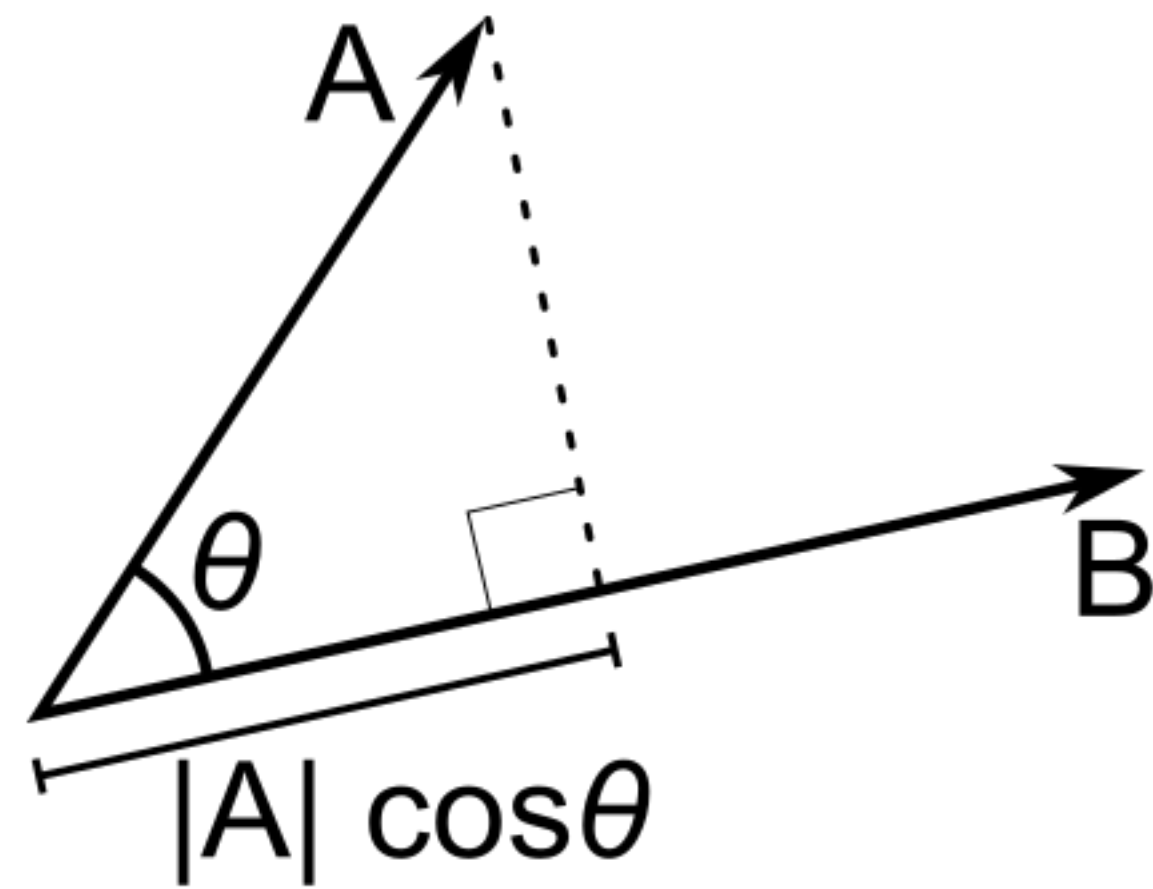
$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}_1 \mathbf{y}_1 + \mathbf{x}_2 \mathbf{y}_2$$

These two definitions are equivalent. Prove it in WA2.

Length (norm) of \mathbf{x}

- Inner product of two vectors tells us how "similar" two vectors are.
 - 0 if they are orthogonal;
 - Maximal if they have the same direction.

Vector Projection



Magnitude of projection of vector **A** in the direction of **B**

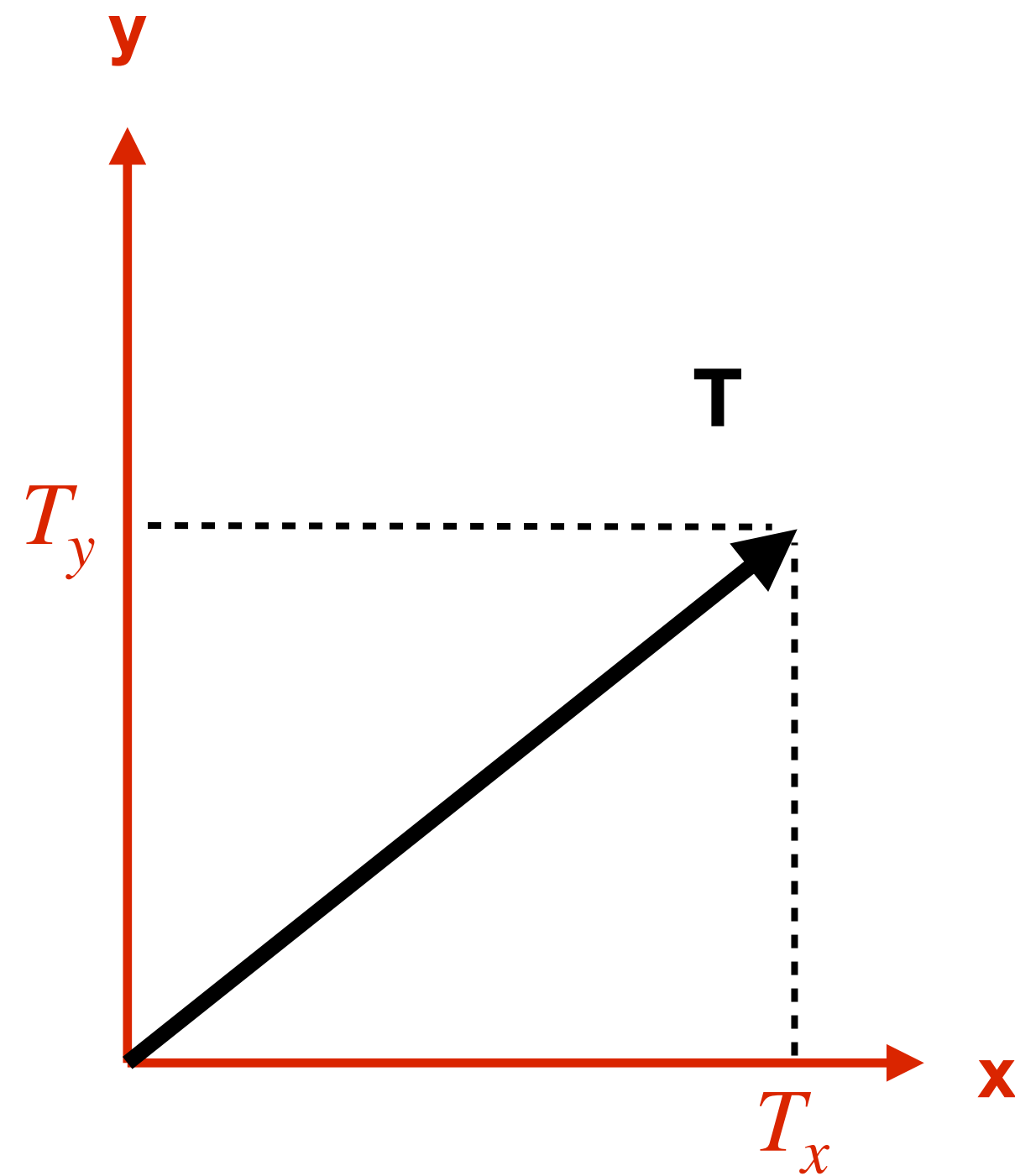
The projected vector

$$\mathbf{A} \cdot \mathbf{B} = \|\mathbf{A}\| \|\mathbf{B}\| \cos \theta$$

$$A_B = \|\mathbf{A}\| \cos \theta = \frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{B}\|}$$

$$A_B \hat{\mathbf{B}} = \frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{B}\|} \frac{\mathbf{B}}{\|\mathbf{B}\|} = \frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{B}\|^2} \mathbf{B}$$

Projecting a Vector to Two Orthogonal Vectors

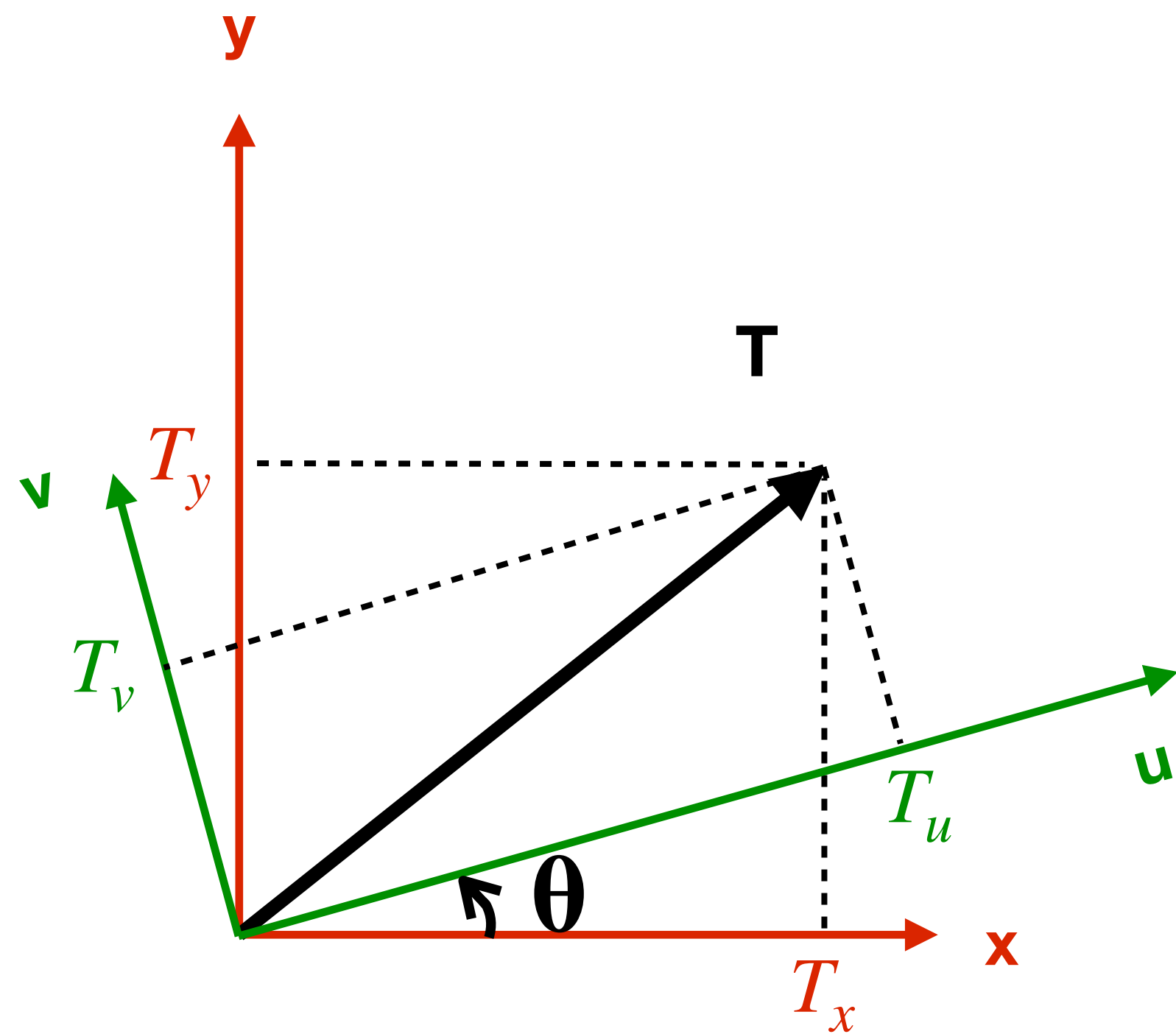


$$T_x = \frac{\mathbf{T} \cdot \mathbf{x}}{\|\mathbf{x}\|}, \quad T_y = \frac{\mathbf{T} \cdot \mathbf{y}}{\|\mathbf{y}\|}, \quad \mathbf{T} = [T_x, T_y]$$

$$\mathbf{T} = T_x \hat{\mathbf{x}} + T_y \hat{\mathbf{y}} = \frac{\mathbf{T} \cdot \mathbf{x}}{\|\mathbf{x}\|^2} \mathbf{x} + \frac{\mathbf{T} \cdot \mathbf{y}}{\|\mathbf{y}\|^2} \mathbf{y}$$

- What's happened here? We can express an arbitrary vector **T** using two orthogonal "basis" vectors by projecting the vector to the two basis vectors.

Change of Coordinates



$$T_x = \frac{\mathbf{T} \cdot \mathbf{x}}{\|\mathbf{x}\|}, \quad T_y = \frac{\mathbf{T} \cdot \mathbf{y}}{\|\mathbf{y}\|}, \quad \mathbf{T} = [T_x, T_y]$$

$$T_u = \frac{\mathbf{T} \cdot \mathbf{u}}{\|\mathbf{u}\|}, \quad T_v = \frac{\mathbf{T} \cdot \mathbf{v}}{\|\mathbf{v}\|}, \quad \mathbf{T} = [T_u, T_v]$$

$$\mathbf{T} = \frac{\mathbf{T} \cdot \mathbf{u}}{\|\mathbf{u}\|^2} \mathbf{u} + \frac{\mathbf{T} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v}$$

$[T_x, T_y]$ and $[T_u, T_v]$ are related by a geometric transformation. WA1 asks you to come up with the transformation.

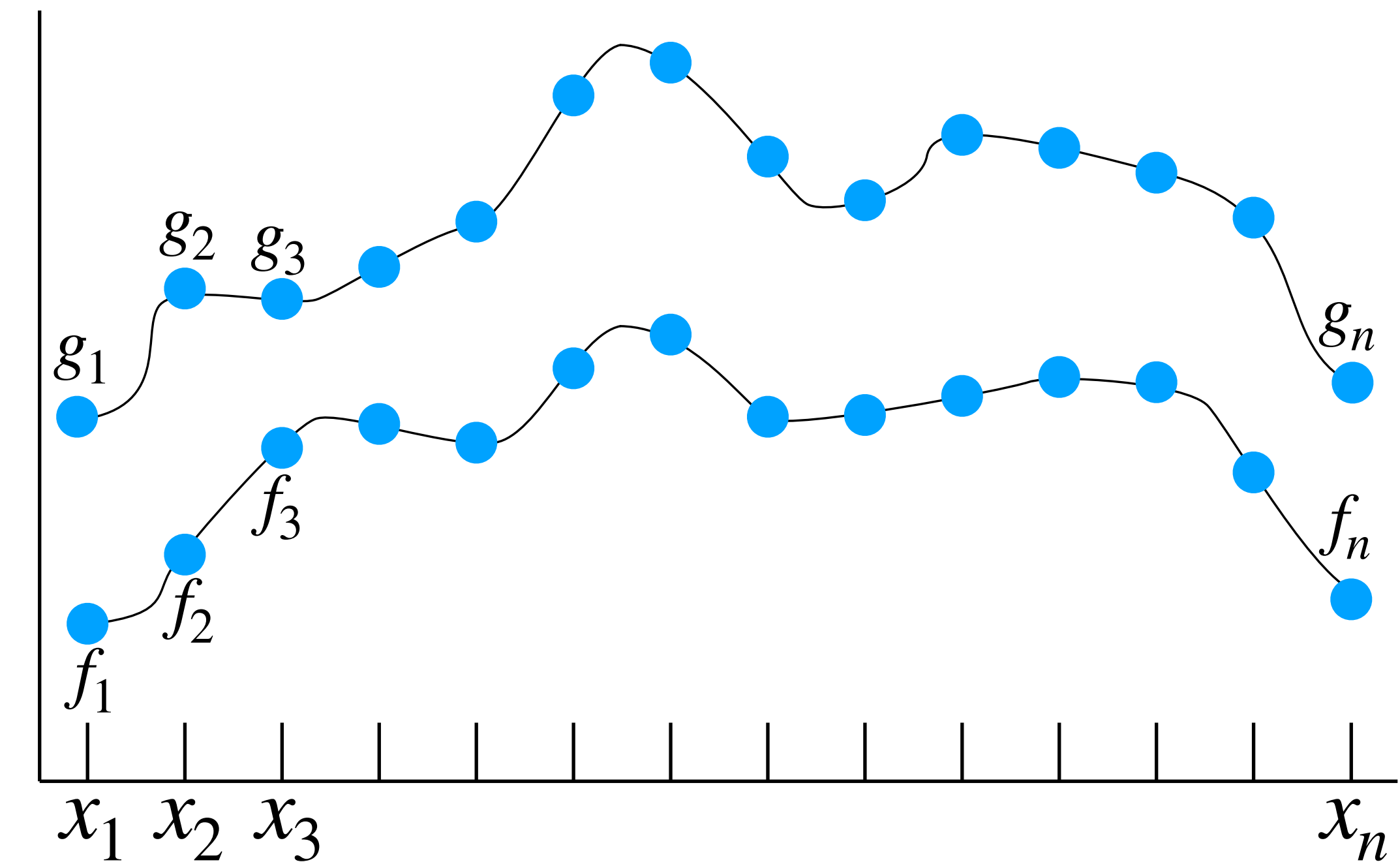
Functions are Vectors

- Assume functions f and g are defined as N discrete points
- f and g are nothing more than two N -dimensional vectors.

$$f = [f_1, f_2, \dots, f_n]$$

$$g = [g_1, g_2, \dots, g_n]$$

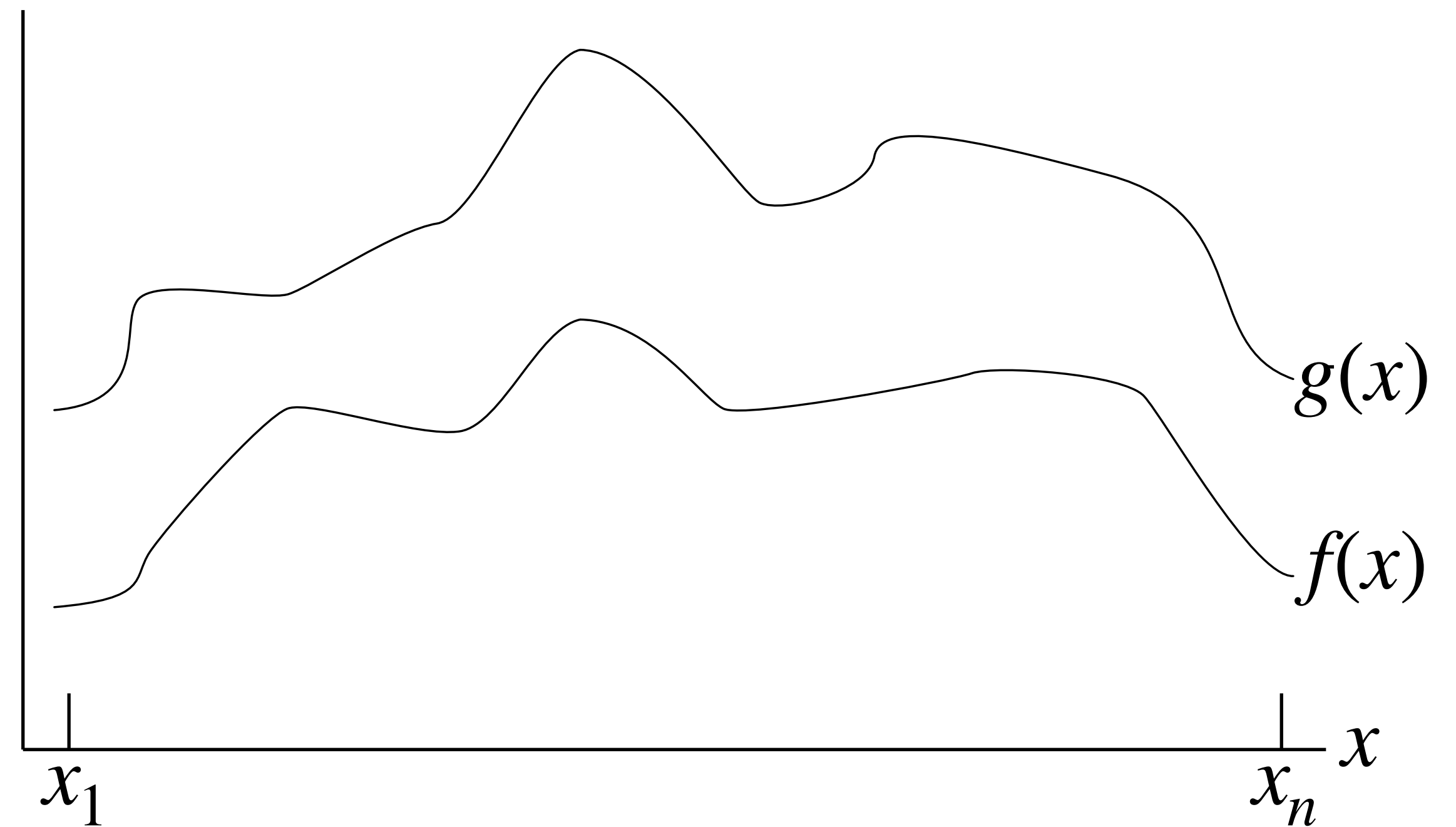
Inner product of two N -dimensional vectors



$$f \cdot g = \sum_{i=1}^n f_i g_i$$

Continuous Function = Infinite-Dimensional Vector

- When f and g are continuous functions, they can be seen as two **infinite-dimensional** vectors.
- The inner product of two continuous functions is the **integration** of the product.
- Use complex conjugate of g if they are complex-valued.



Inner product of two
continuous functions

$$f \cdot g = \int_{x_1}^{x_n} f(x)g(x)dx$$

Function Inner Product

Inner product of two continuous functions

$$f \cdot g = \int_{x_1}^{x_n} f(x)g(x)dx$$

Function f and g are orthogonal if:

$$f \cdot g = \int_{x_1}^{x_n} f(x)g(x)dx = 0$$

Norm of a function g is:

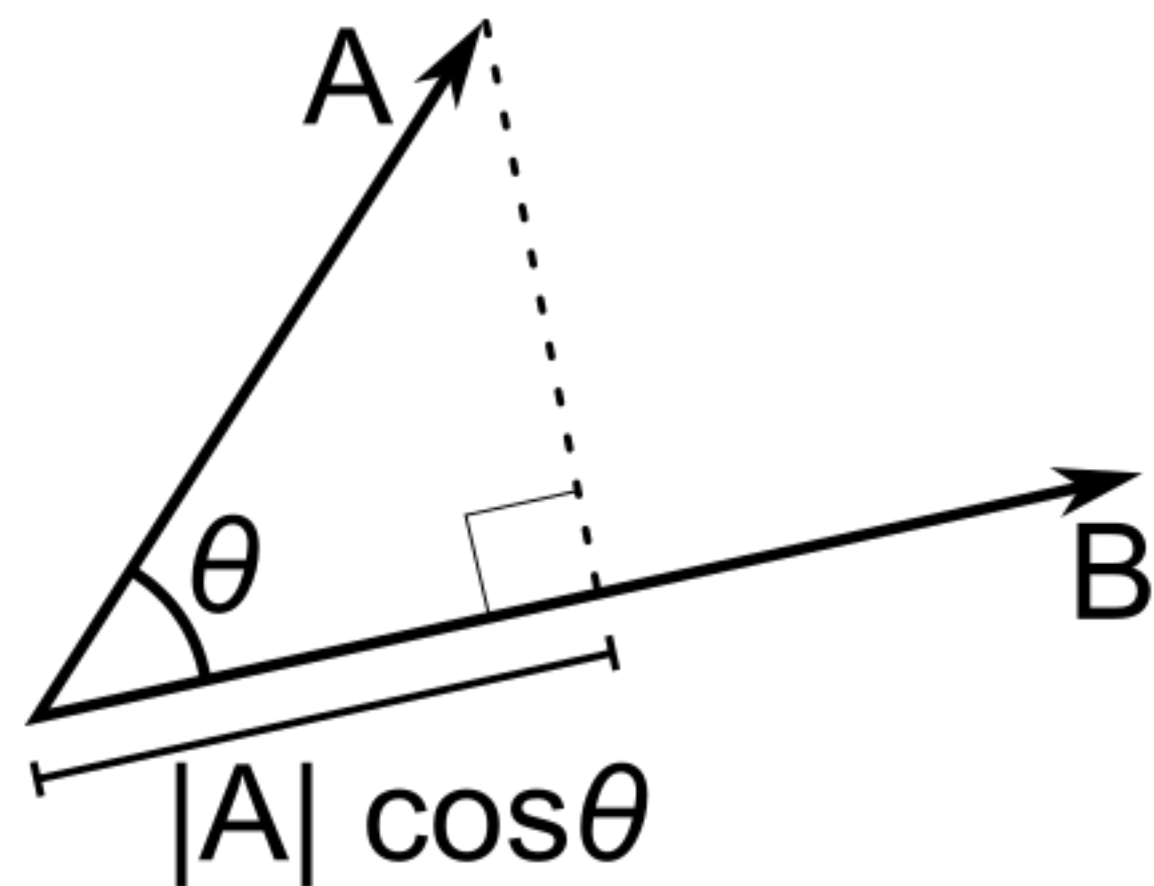
$$\|g\| = \left(\int_{x_1}^{x_n} g(x)^2 dx \right)^{1/2}$$

- An infinite-dimensional space that also defines vector operations (e.g., dot product) is called a *Hilbert space*.

Vector vs. Function Projection

Vector A projected to
vector B

$$A_B \hat{\mathbf{B}} = \frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{B}\|^2} \mathbf{B}$$



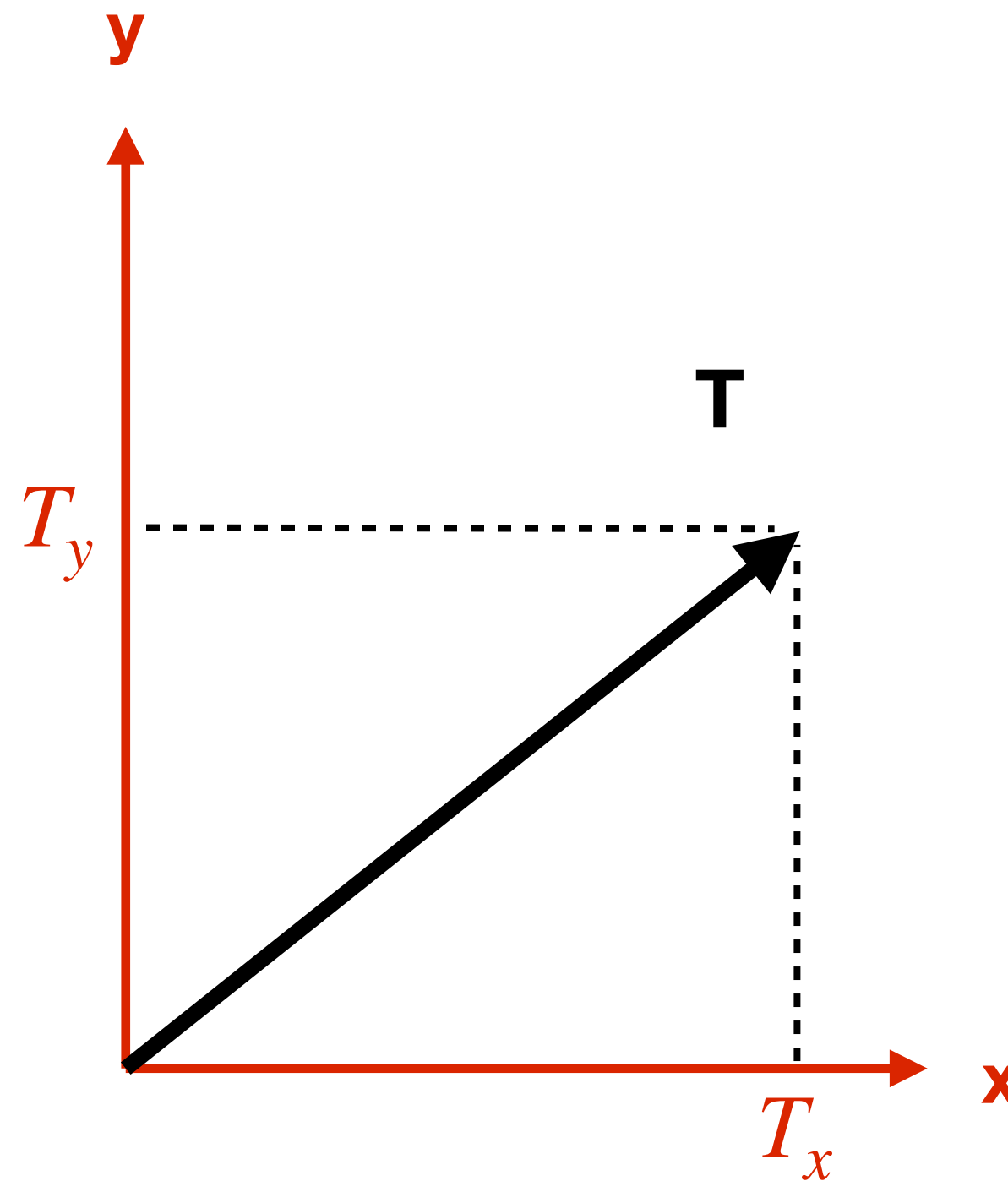
Function f projected to
function g

$$f_g = \frac{f \cdot g}{\|g\|^2} g$$

Recall: Expressing a Vector Using Basis Vectors

- We can express a 2D vector as a sum of two orthogonal basis vectors.
 - By projecting \mathbf{T} to the two basis vectors

$$\mathbf{T} = \frac{\mathbf{T} \cdot \mathbf{x}}{\|\mathbf{x}\|^2} \mathbf{x} + \frac{\mathbf{T} \cdot \mathbf{y}}{\|\mathbf{y}\|^2} \mathbf{y}$$



Expressing a Function as Basis Functions

- Similarly, we can express a function using “basis functions”.
 - How many basis functions do we need?
- Expressing a continuous function requires infinitely many basis functions.
 - Basis functions must be orthogonal.

$$\mathbf{T} = \frac{\mathbf{T} \cdot \mathbf{x}}{\|\mathbf{x}\|^2} \mathbf{x} + \frac{\mathbf{T} \cdot \mathbf{y}}{\|\mathbf{y}\|^2} \mathbf{y}$$

$$f = \sum_{i=1}^{\infty} \frac{f \cdot v_i}{\|v_i\|^2} v_i$$

v_1, v_2, \dots are orthogonal functions

Problem Setup

- Now assuming we have a continuous function f defined over $[t, t+L)$
- How do we represent f as a sum of a set of basis functions?
 - The basis functions need to be defined over $[t, t+L)$ too.

$$\mathbf{T} = \frac{\mathbf{T} \cdot \mathbf{x}}{\|\mathbf{x}\|^2} \mathbf{x} + \frac{\mathbf{T} \cdot \mathbf{y}}{\|\mathbf{y}\|^2} \mathbf{y}$$

$$f = \sum_{i=1}^{\infty} \frac{f \cdot v_i}{\|v_i\|^2} v_i$$

v_1, v_2, \dots are orthogonal functions

A Special Set of Orthogonal Basis Functions

$$\left\{ \cos\left(\frac{2\pi kx}{L}\right) \right\}_{k=0}^{\infty}, \left\{ \sin\left(\frac{2\pi kx}{L}\right) \right\}_{k=1}^{\infty}, \quad k \in \mathbb{Z} \quad \text{mutually orthogonal functions on any } [t, t+L) \text{ interval}$$

$$\cos\left(\frac{0\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \cos\left(\frac{4\pi x}{L}\right), \cos\left(\frac{6\pi x}{L}\right), \dots, \sin\left(\frac{2\pi x}{L}\right), \sin\left(\frac{4\pi x}{L}\right), \sin\left(\frac{6\pi x}{L}\right), \dots$$

For instance:

$$\int_t^{t+L} \cos\left(\frac{2\pi x}{L}\right) \sin\left(\frac{6\pi x}{L}\right) dx = 0$$
$$\int_t^{t+L} \cos\left(\frac{2\pi x}{L}\right) \cos\left(\frac{6\pi x}{L}\right) dx = 0$$

The sine series starts from $k=1$, since $k=0$ leads to a function that's 0 everywhere, i.e. a 0-norm vector.

Fourier Series

$$f = \sum_{i=1}^{\infty} \frac{f \cdot v_i}{\|v_i\|^2} v_i \quad v_i \in \left\{ \left\{ \cos\left(\frac{2\pi kx}{L}\right) \right\}_{k=0}^{\infty}, \left\{ \sin\left(\frac{2\pi kx}{L}\right) \right\}_{k=1}^{\infty}, x \in [t, t+L) \right\}$$

$$f(x) = \sum_{k=0}^{\infty} \left(a_k \cos\left(\frac{2\pi kx}{L}\right) \right) + \sum_{k=1}^{\infty} \left(b_k \sin\left(\frac{2\pi kx}{L}\right) \right), \quad x \in [t, t+L)$$

$$a_k = \frac{f(x) \cdot \left(\cos\left(\frac{2\pi kx}{L}\right) \right)}{\left\| \cos\left(\frac{2\pi kx}{L}\right) \right\|^2} = \frac{2}{L} \int_t^{t+L} f(x) \cos\left(\frac{2\pi kx}{L}\right) dx, \quad \text{if } k \neq 0$$

$$b_k = \frac{f(x) \cdot \left(\sin\left(\frac{2\pi kx}{L}\right) \right)}{\left\| \sin\left(\frac{2\pi kx}{L}\right) \right\|^2} = \frac{2}{L} \int_t^{t+L} f(x) \sin\left(\frac{2\pi kx}{L}\right) dx$$

$$= \frac{1}{L} \int_t^{t+L} f(x) \cos\left(\frac{2\pi kx}{L}\right) dx, \quad \text{if } k = 0$$

Fourier Coefficients

- Rearrange terms to get a more compact equation.
- a_k and b_k are called the **Fourier coefficients** of $f(x)$.

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{2\pi kx}{L}\right) + b_k \sin\left(\frac{2\pi kx}{L}\right) \right), \quad x \in [t, t+L), \quad k \in \mathbb{Z}^+$$

$$a_k = \frac{2}{L} \int_t^{t+L} f(x) \cos\left(\frac{2\pi kx}{L}\right) dx \qquad b_k = \frac{2}{L} \int_t^{t+L} f(x) \sin\left(\frac{2\pi kx}{L}\right) dx$$

Fourier Coefficients

- Rearrange terms to get a more compact equation.
- a_k and b_k are called the **Fourier coefficients** of $f(x)$.

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(\omega_k x) + b_k \sin(\omega_k x)) \quad \omega_k = \frac{2k\pi}{L}$$

$$a_k = \frac{2}{L} \int_0^L f(x) \cos(\omega_k x) dx \quad b_k = \frac{2}{L} \int_0^L f(x) \sin(\omega_k x) dx$$

Frequency

- As k increases, the frequency of the basis functions increase.

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(\omega_k x) + b_k \sin(\omega_k x)) \quad \omega_k = \frac{2k\pi}{L}$$

$$a_k = \frac{2}{L} \int_0^L f(x) \cos(\omega_k x) dx \quad b_k = \frac{2}{L} \int_0^L f(x) \sin(\omega_k x) dx$$

Frequency-Domain Representation of a Function

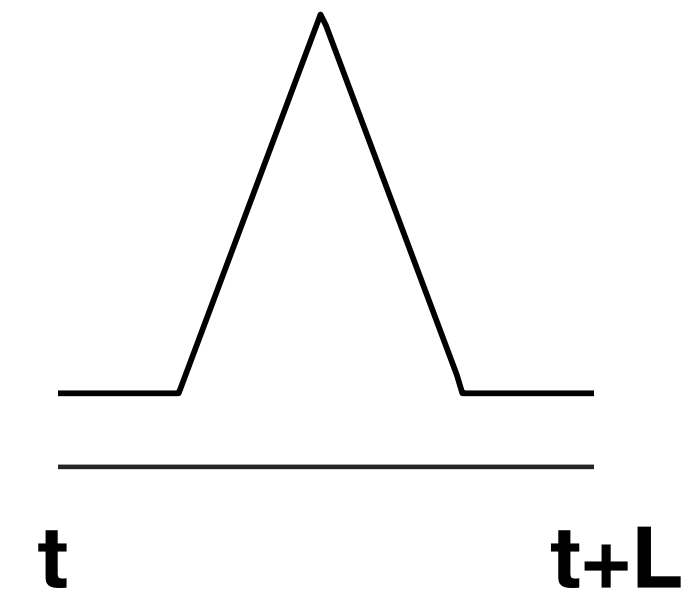
- We have converted a function $f(x)$ into an infinite sequence of a_k and b_k .
- The (infinite) coefficient sequence is called the frequency-domain representation, or the **spectrum**, of the function f .

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(\omega_k x) + b_k \sin(\omega_k x)) \quad \omega_k = \frac{2k\pi}{L}$$

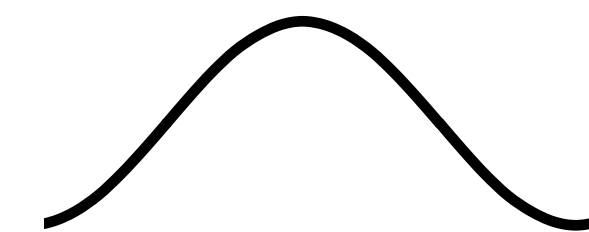
$$a_k = \frac{2}{L} \int_0^L f(x) \cos(\omega_k x) dx \quad b_k = \frac{2}{L} \int_0^L f(x) \sin(\omega_k x) dx$$

Applies to Only Periodic Functions

- We assumed f and the basis functions are defined only over a fixed interval $[t, t+L)$
- But sinusoidal functions have infinite span; they are not limited to a fixed interval
- What if we just add the Fourier series over the entire span?
- We will generate a **L-periodic** function!

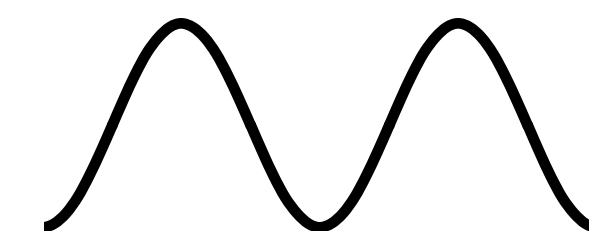


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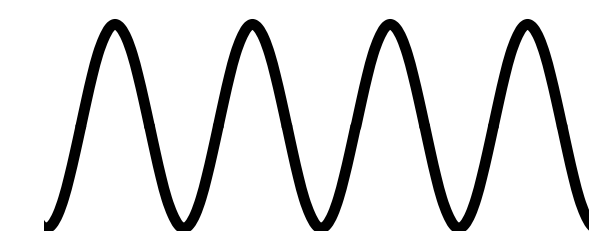
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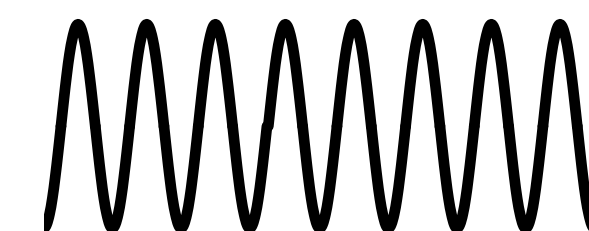
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Fourier Series: What It Says



Joseph Fourier (1768 – 1830)

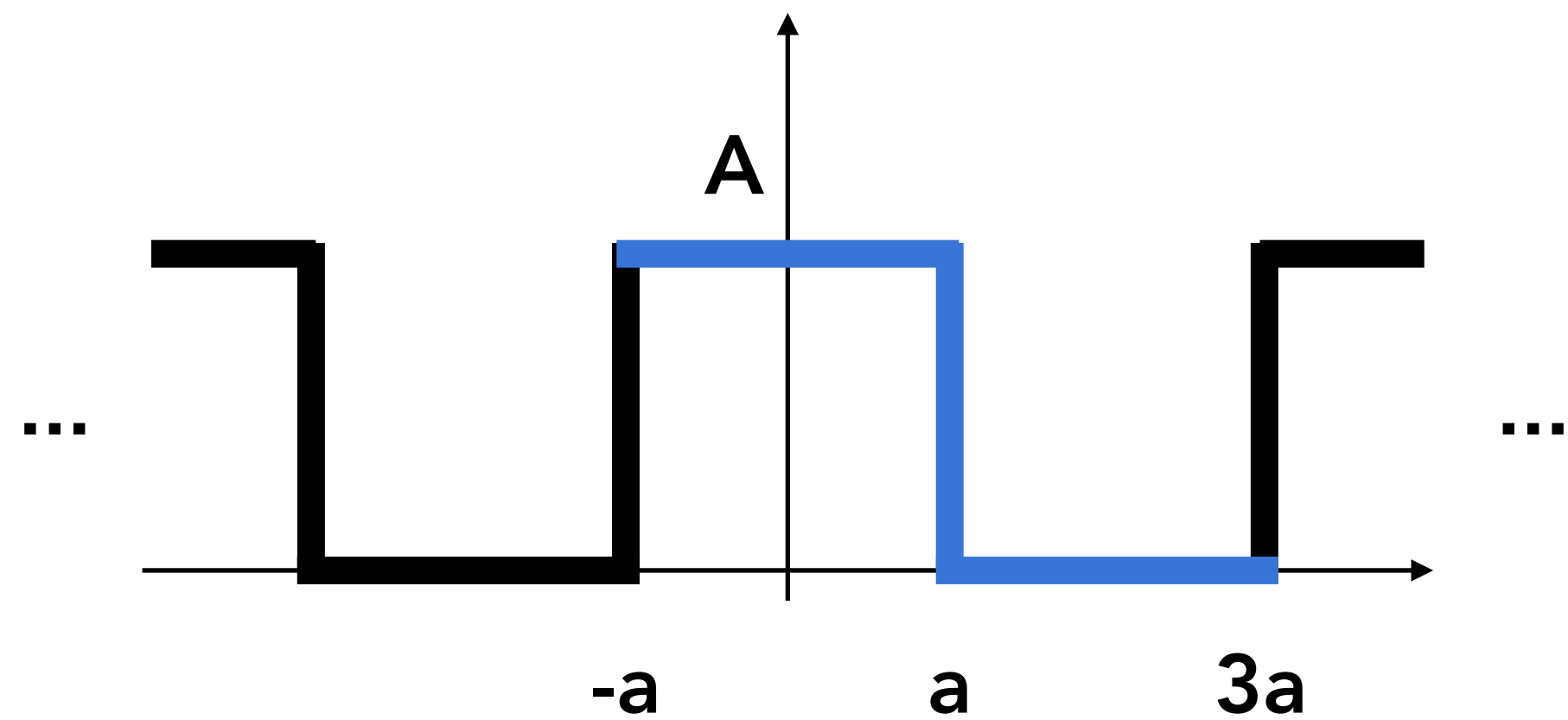
Any **periodic** function can be represented as an infinite sum of **weighted sines and cosines** with increasing frequencies.

Fourier coefficients

Basis functions

Back To the First Example

$$\omega_k = \frac{2k\pi}{L}$$



A square function

Period $L = 4a$

Frequency is $1/4a$

Amplitude is A

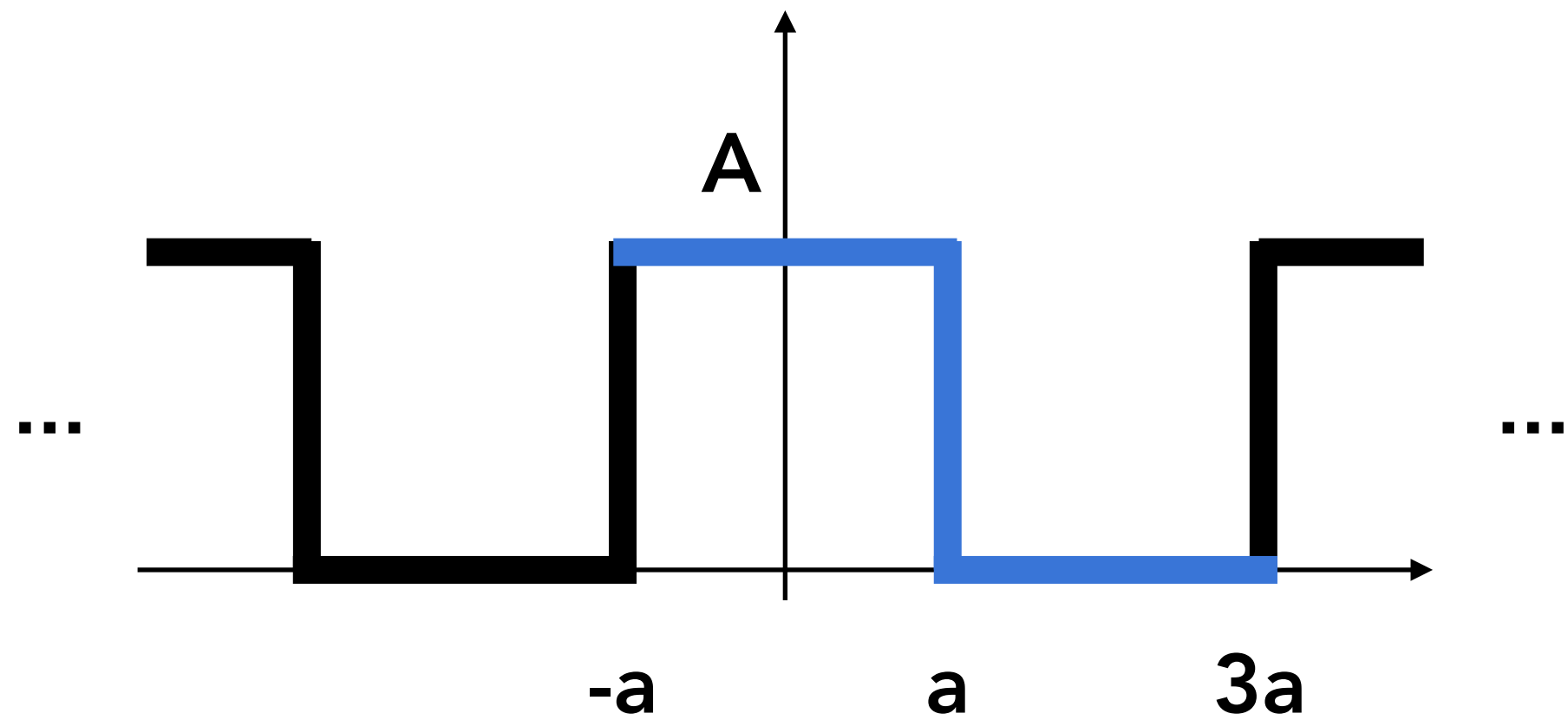
$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(\omega_k x) + b_k \sin(\omega_k x))$$

$$a_k = \frac{2}{L} \int_t^{t+L} f(x) \cos(\omega_k x) dx$$

$$b_k = \frac{2}{L} \int_t^{t+L} f(x) \sin(\omega_k x) dx$$

$$f(x) = \frac{A}{2} + \frac{2A}{\pi} \cos\left(\frac{1}{4a}x\right) + \frac{-2A}{3\pi} \cos\left(\frac{3}{4a}x\right) + \frac{2A}{5\pi} \cos\left(\frac{5}{4a}x\right) + \frac{-2A}{7\pi} \cos\left(\frac{7}{4a}x\right) + \dots$$

Back To the First Example



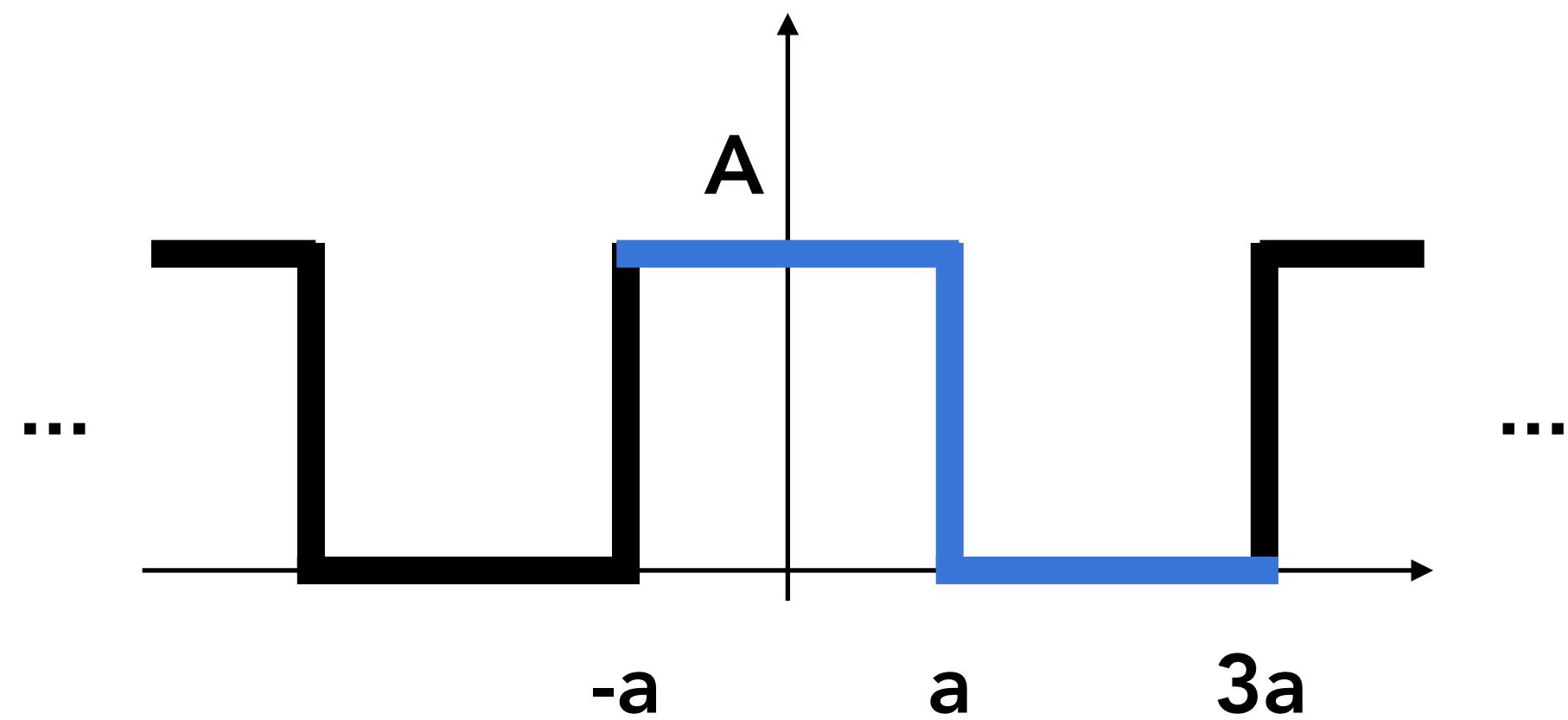
$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{k\pi}{2a}x\right) + b_k \sin\left(\frac{k\pi}{2a}x\right) \right)$$

$$a_k = \frac{1}{2a} \int_{-a}^{3a} f(x) \cos\left(\frac{k\pi}{2a}x\right) dx$$

$$b_k = \frac{1}{2a} \int_{-a}^{3a} f(x) \sin\left(\frac{k\pi}{2a}x\right) dx$$

$$f(x) = \frac{A}{2} + \frac{2A}{\pi} \cos\left(\frac{1}{4a}x\right) + \frac{-2A}{3\pi} \cos\left(\frac{3}{4a}x\right) + \frac{2A}{5\pi} \cos\left(\frac{5}{4a}x\right) + \frac{-2A}{7\pi} \cos\left(\frac{7}{4a}x\right) + \dots$$

Back To the First Example



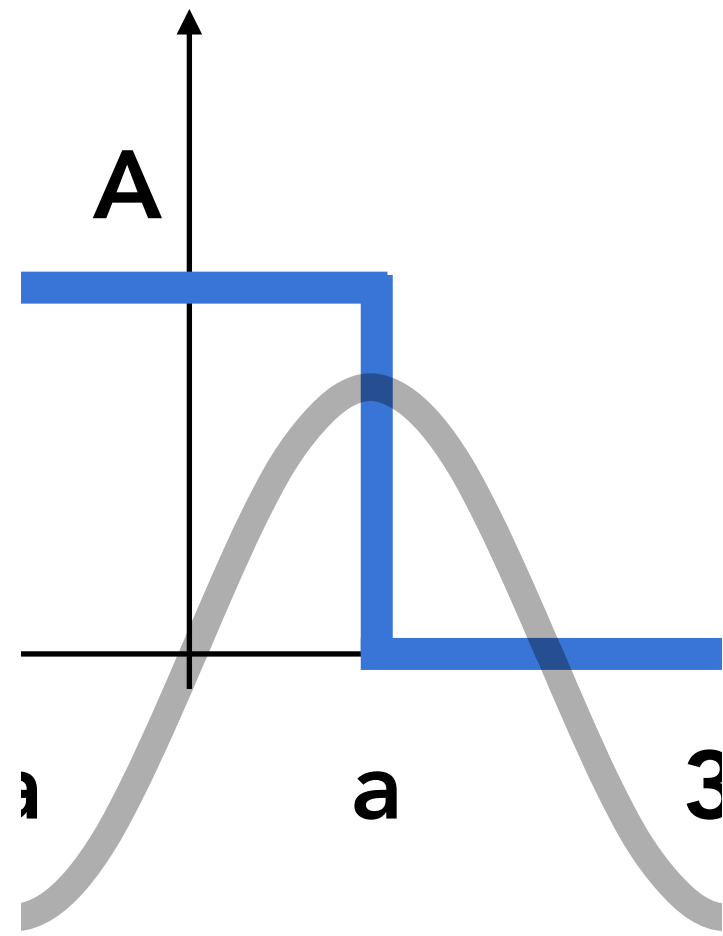
$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{k\pi}{2a}x\right) + b_k \sin\left(\frac{k\pi}{2a}x\right) \right)$$

$$a_k = \frac{1}{2a} \int_{-a}^{3a} f(x) \cos\left(\frac{k\pi}{2a}x\right) dx$$

$$a_0 = \frac{1}{2a} \int_{-a}^{3a} f(x) \cos\left(\frac{0\pi}{2a}x\right) dx = \frac{1}{2a} \int_{-a}^{3a} f(x) dx = \frac{1}{2a} 2aA = A$$

$$f(x) = \frac{A}{2} + \frac{2A}{\pi} \cos\left(\frac{1}{4a}x\right) + \frac{-2A}{3\pi} \cos\left(\frac{3}{4a}x\right) + \frac{2A}{5\pi} \cos\left(\frac{5}{4a}x\right) + \frac{-2A}{7\pi} \cos\left(\frac{7}{4a}x\right) + \dots$$

Why No *Sine* Term?

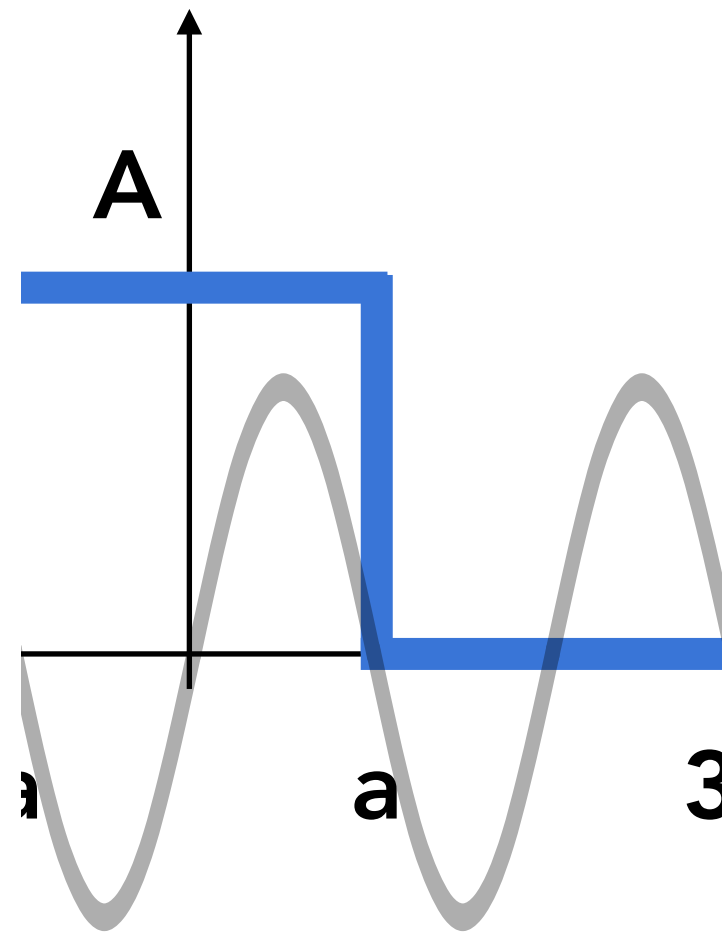


$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{k\pi}{2a}x\right) + b_k \sin\left(\frac{k\pi}{2a}x\right) \right)$$

$$= \frac{1}{2a} \int_{-a}^{3a} f(x) \sin\left(\frac{k\pi}{2a}x\right) dx$$

- The period of the *sine* functions are $4a/k$, i.e., $4a, 2a, a, a/2, \dots$

Why No *Sine* Term?



$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{k\pi}{2a}x\right) + b_k \sin\left(\frac{k\pi}{2a}x\right) \right)$$

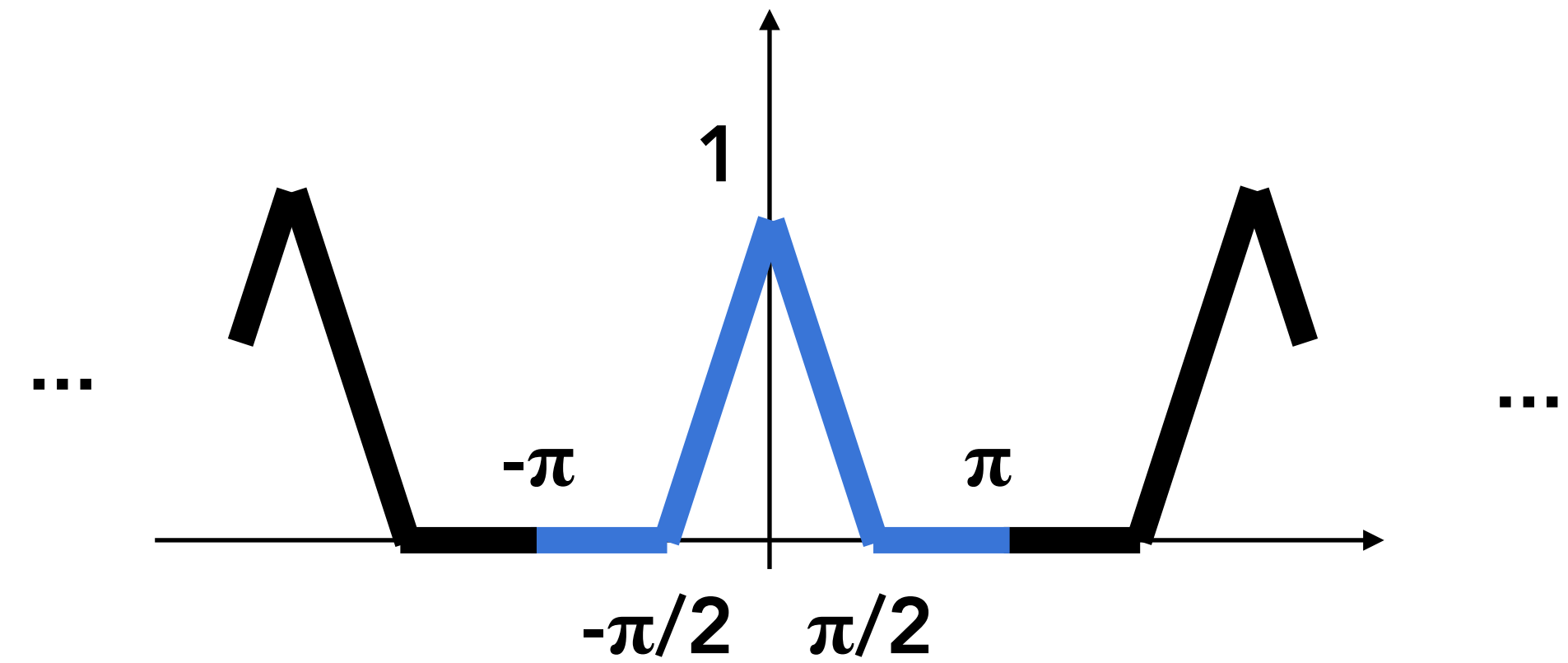
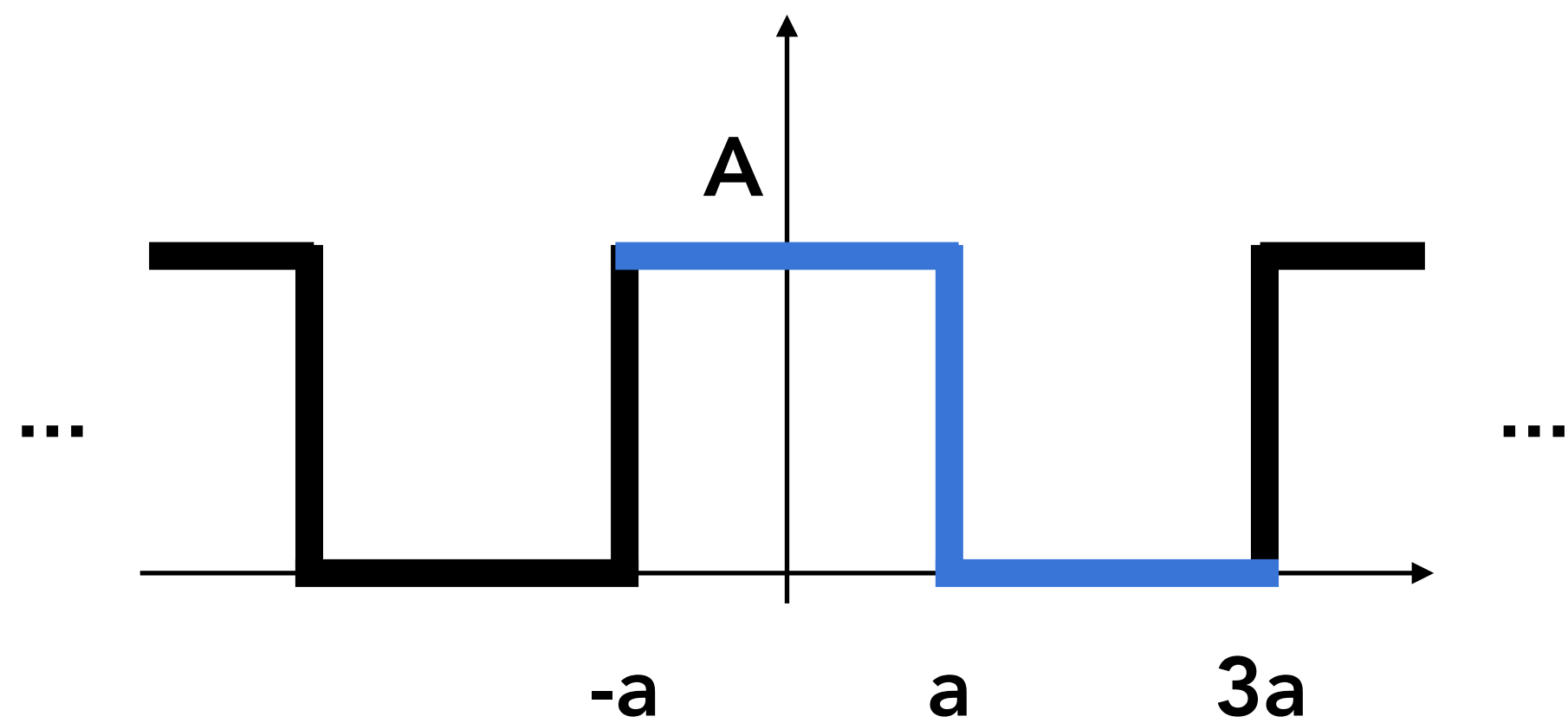
$$= \frac{1}{2a} \int_{-a}^{3a} f(x) \sin\left(\frac{k\pi}{2a}x\right) dx$$

- The period of the *sine* functions are $4a/k$, i.e., $4a, 2a, a, a/2, \dots$
- Since *sine* is an odd function, b_k always becomes 0

$$f(x) = \frac{A}{2} + \frac{2A}{\pi} \cos\left(\frac{1}{4a}x\right) + \frac{-2A}{3\pi} \cos\left(\frac{3}{4a}x\right) + \frac{2A}{5\pi} \cos\left(\frac{5}{4a}x\right) + \frac{-2A}{7\pi} \cos\left(\frac{7}{4a}x\right) + \dots$$

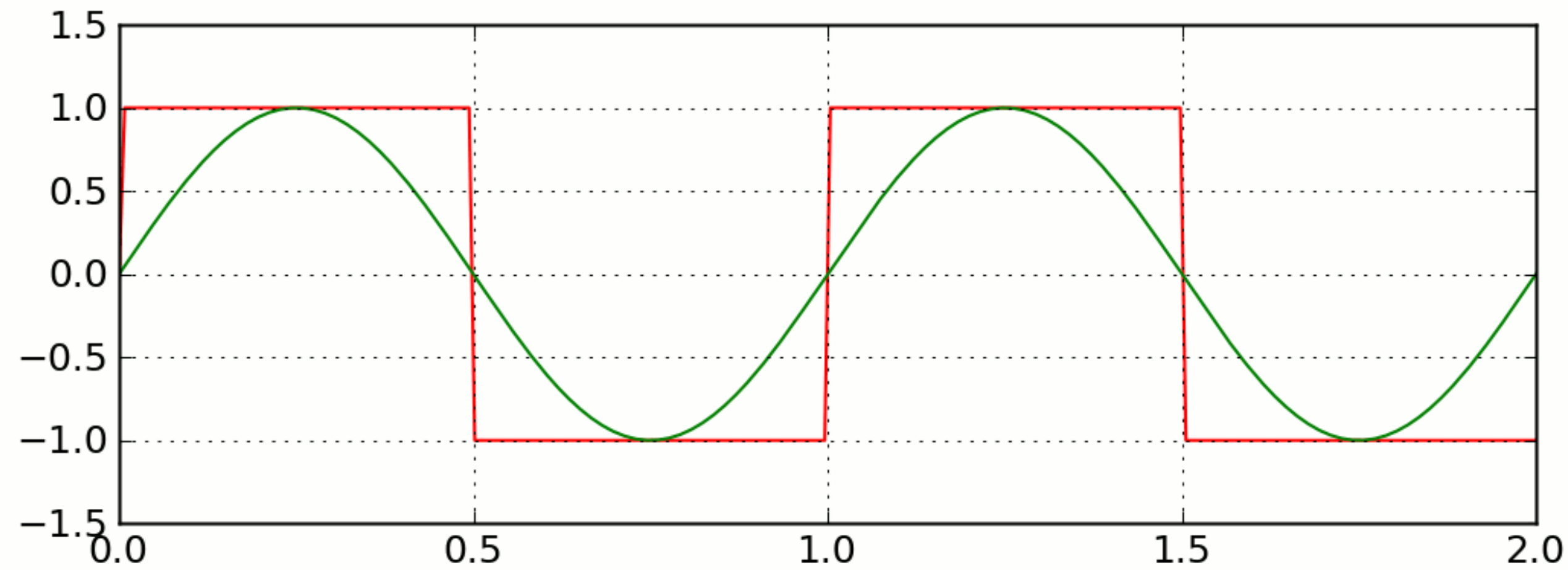
Even Functions Need only Cosine Terms

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2} \cos(x) + \frac{4}{(3\pi)^2} \cos(3x) + \frac{4}{(5\pi)^2} \cos(5x) + \frac{4}{(7\pi)^2} \cos(7x) + \dots$$



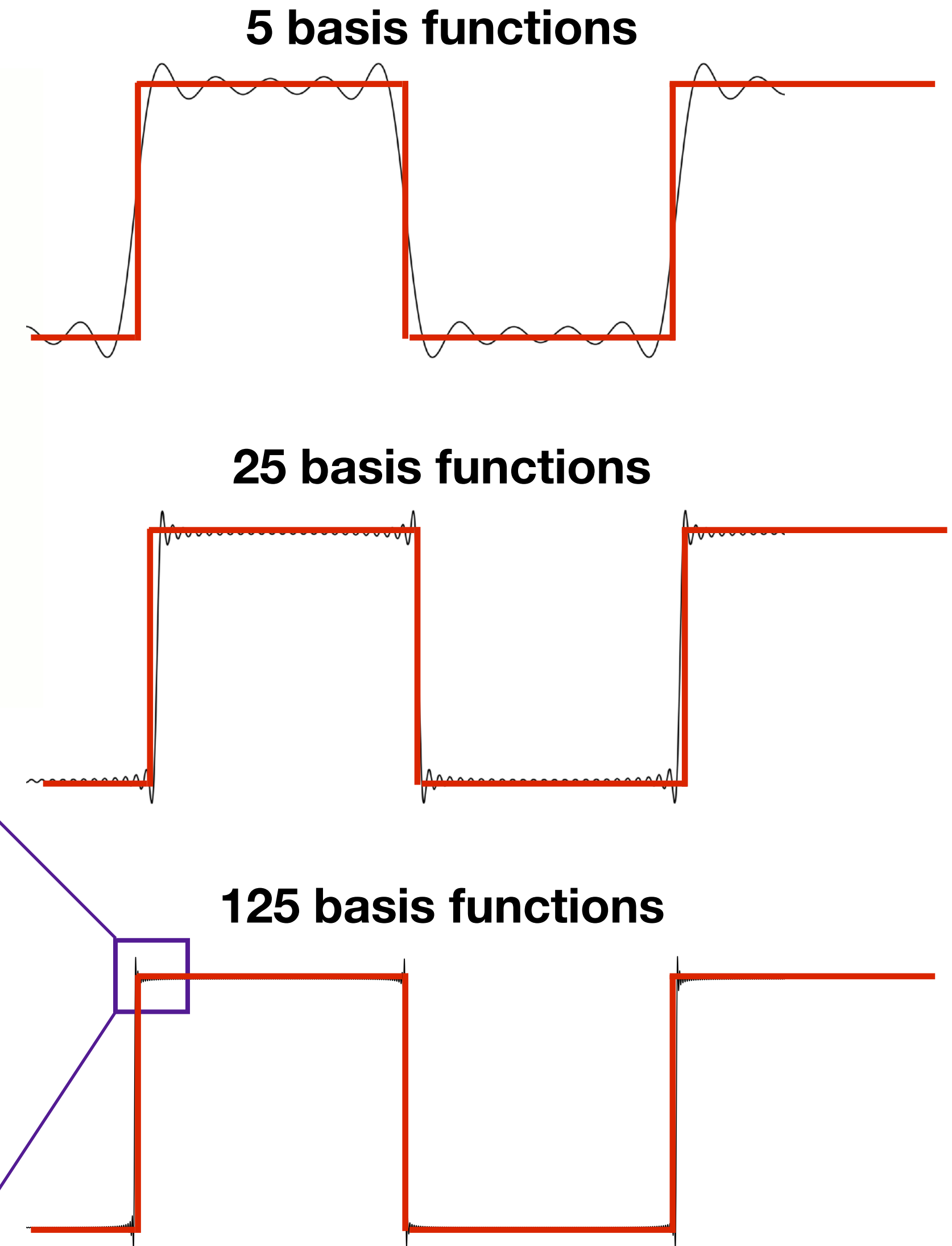
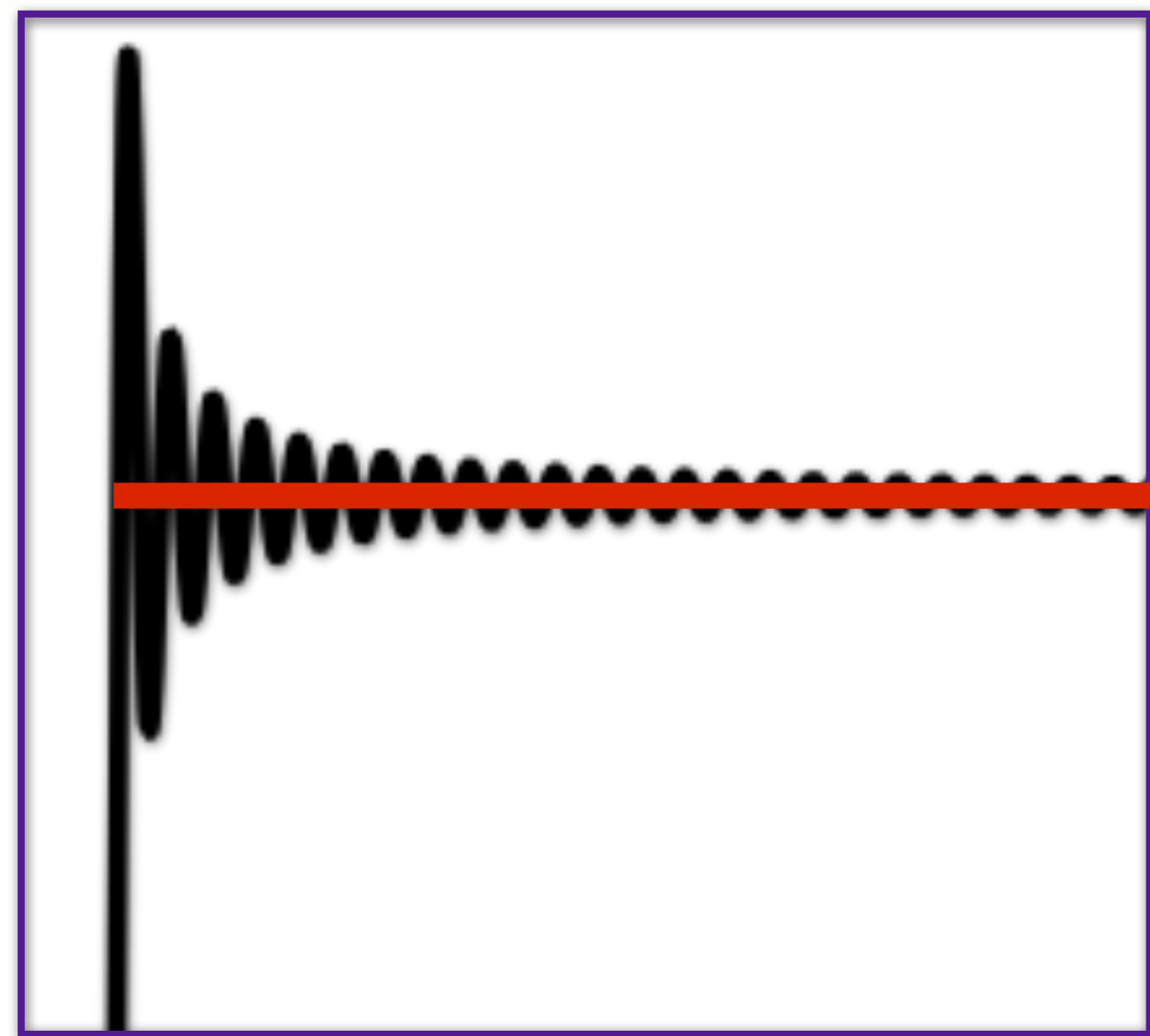
$$f(x) = \frac{A}{2} + \frac{2A}{\pi} \cos\left(\frac{1}{4a}x\right) + \frac{-2A}{3\pi} \cos\left(\frac{3}{4a}x\right) + \frac{2A}{5\pi} \cos\left(\frac{5}{4a}x\right) + \frac{-2A}{7\pi} \cos\left(\frac{7}{4a}x\right) + \dots$$

Gibbs Phenomena (Ringing)



Sudden change of value in a signal requires extremely high-frequency signals to represent.

We say the signal "contains **high-frequency** components."



Complex Basis Functions

Complex Basis Functions

$$\left\{ e^{\frac{ik\pi x}{L}} \right\}_{k=-\infty}^{k=\infty}, \quad x \in [-L, L), \quad k \in \mathbb{Z} \quad \left\{ \dots, e^{\frac{-2i\pi x}{L}}, e^{\frac{-i\pi x}{L}}, e^{\frac{i0\pi x}{L}}, e^{\frac{i1\pi x}{L}}, \dots \right\}$$

$$\phi_k = e^{\frac{ik\pi x}{L}} \quad \phi_i \cdot \phi_j = \begin{cases} 0, & \text{if } i \neq j, \\ 2L, & \text{if } i = j. \end{cases}$$

Mutually orthogonal vectors over $[-L, L)$ with a norm of $\sqrt{2L}$

Complex Fourier Series

- Can prove equivalence using Euler's formula: $e^{ikx} = \cos(kx) + i \sin(kx)$
- More general and concise

$$\phi_k = e^{\frac{ik\pi x}{L}}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \phi_k$$

$$c_k = \frac{f(x) \cdot \phi_k}{\|\phi_k\|^2} = \frac{1}{2L} \int_{-L}^L f(x) \bar{\phi}_k dx = \frac{1}{2L} \int_{-L}^L f(x) e^{-\frac{ik\pi x}{L}} dx$$

Geometric Interpretation

- $\{c_k\}$ is a Hilbert-space vector transformed from f , another Hilbert-space vector.
- What kind of transformation is this?

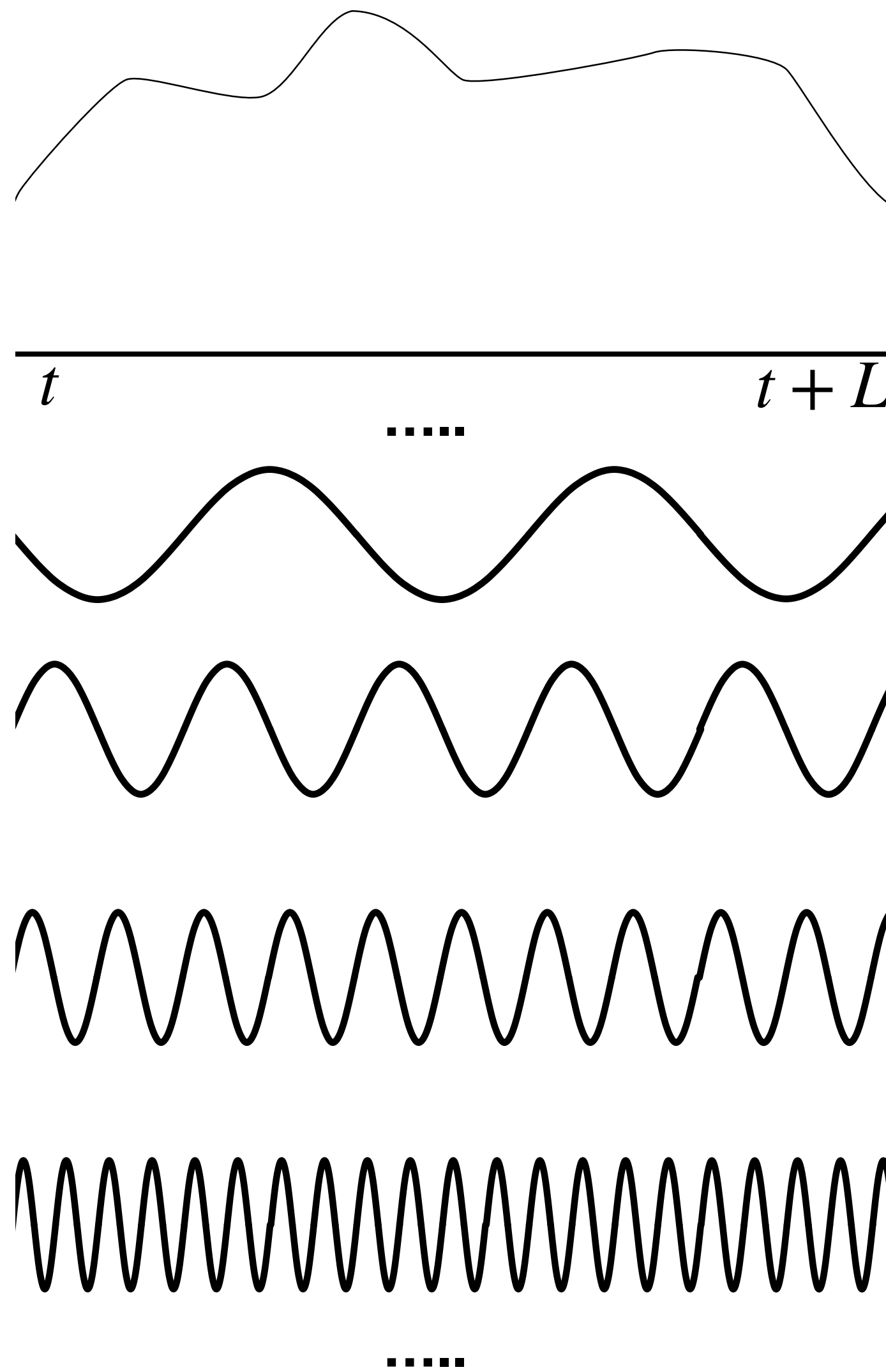
$$\phi_k = e^{\frac{ik\pi x}{L}}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \phi_k$$

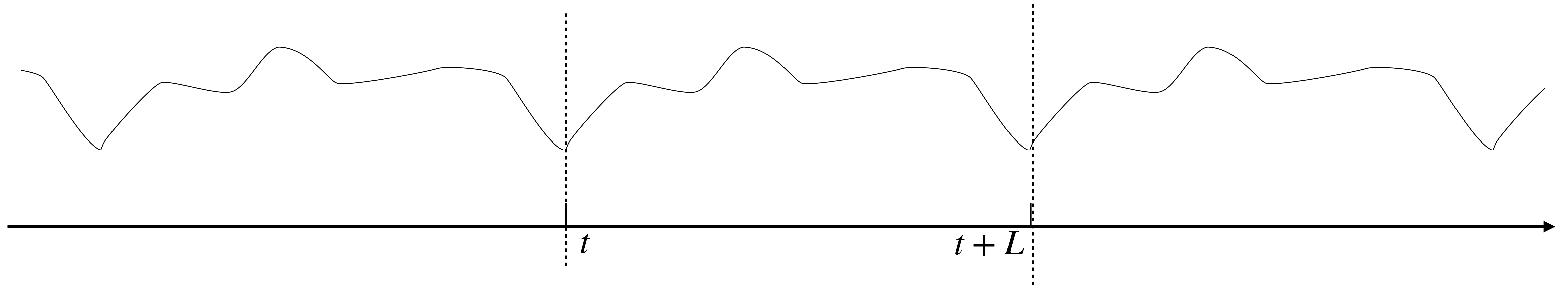
$$\begin{bmatrix} | & | & | & | & | \\ \cdots & \phi_{k-1} & \phi_k & \phi_{k+1} & \cdots \\ | & | & | & | & | \end{bmatrix} \times \begin{bmatrix} | \\ c_{k-1} \\ c_k \\ c_{k+1} \\ | \end{bmatrix} = \begin{bmatrix} | \\ f \\ | \end{bmatrix}$$

Discrete Fourier Transform (DFT)

Recall: Fourier Series

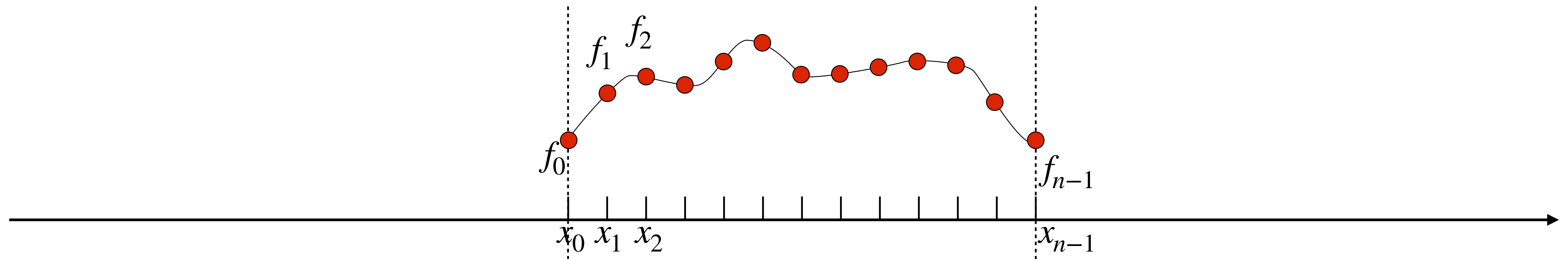


Recall: Fourier Series



- **Fourier series applies to:**
 - Finite interval (say L) + Continuous function
 - Or equivalently: an L -periodic function with an infinite span
- **We need infinitely many basis functions, since a continuous function is an infinitely-dimensional vector**

Real-World Scenarios Impose Restrictions



- In real-world applications, the signals we obtain are discrete samples over a finite interval
 - For instance, audio files are usually 44.1 kHz, i.e., 44,100 samples per second
- We need a discrete form of Fourier series
- Given a sequence of n points, express it as a sum of other sequences
 - This time, we need n other sequences

Discrete Complex Basis Functions

$\left\{ e^{\frac{i2\pi jk}{n}} \right\}, \quad j \in [0, 1, \dots, n-1]$ are n mutually orthogonal n -dimensional vectors

$\left\{ \dots, e^{\frac{i0k}{n}}, e^{\frac{i2\pi k}{n}}, e^{\frac{i4\pi k}{n}}, e^{\frac{i6\pi k}{n}}, \dots \right\} \quad k \in [0, 1, \dots, n-1]$

For instance:
$$\sum_{k=0}^{n-1} e^{\frac{i2\pi k}{n}} e^{\frac{i4\pi k}{n}} = 0$$

Discrete Fourier Transform (Series)

$$f_k = \frac{1}{n} \sum_{j=0}^{n-1} \hat{f}_k e^{\frac{i2\pi jk}{n}} \quad \hat{f}_k = \sum_{j=0}^{n-1} f_j e^{\frac{-i2\pi jk}{n}} \quad k \in [0, 1, \dots, n-1]$$

- This is called Discrete Fourier Transform
 - But it's a misnomer; should really be called Discrete Fourier Series
- Fast Fourier Transform (FFT) is a fast way to calculate DFT
- Project to the basis vectors and calculate the DFT coefficients
- Note that these basis functions are not a subset of the those used in the continuous case!

DFT Matrix

$$f_k = \frac{1}{n} \sum_{j=0}^{n-1} \hat{f}_k \omega_n^{-jk} \quad \hat{f}_k = \sum_{j=0}^{n-1} f_j \omega_n^{jk} \quad \omega_n = e^{\frac{-i2\pi}{n}}$$

The DFT Matrix

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \dots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)^2} \end{bmatrix} \times \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-2} \\ f_{n-1} \end{bmatrix} = \begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \vdots \\ \hat{f}_{n-2} \\ \hat{f}_{n-1} \end{bmatrix}$$

DFT

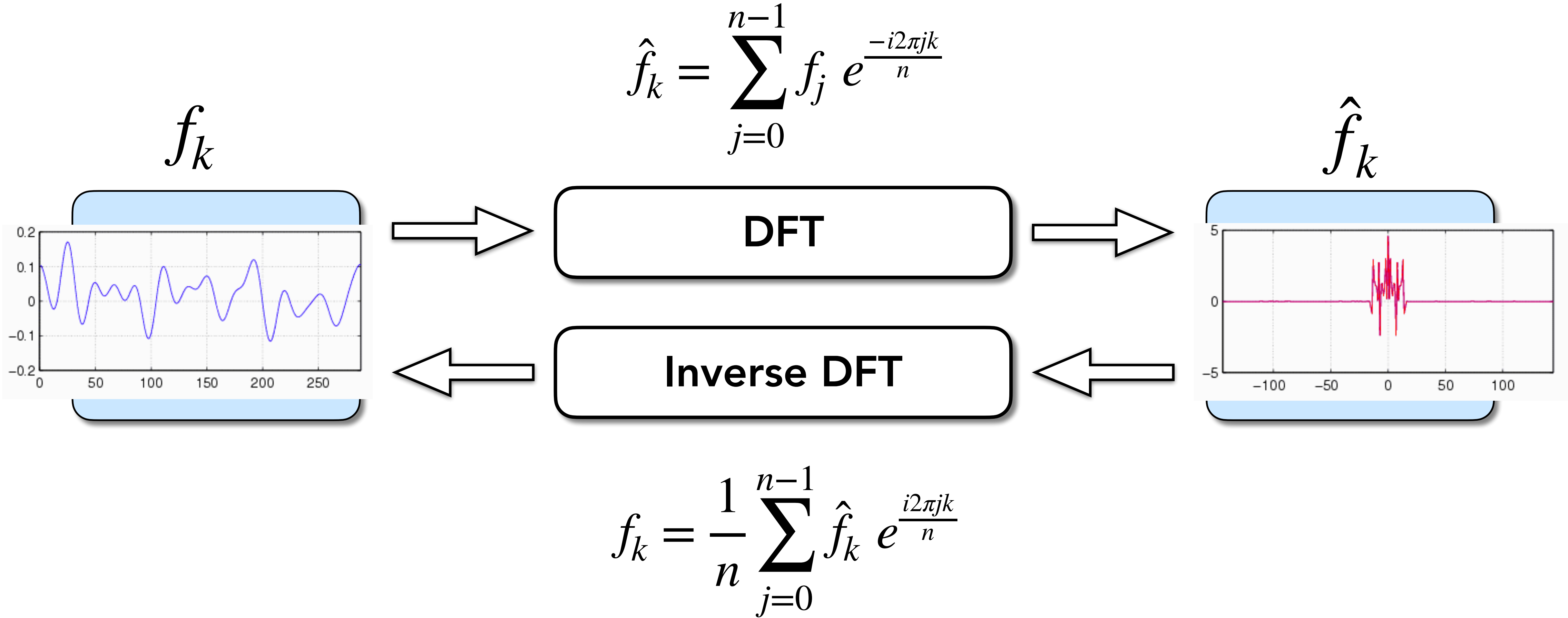
IDFT

DFT Interpretation

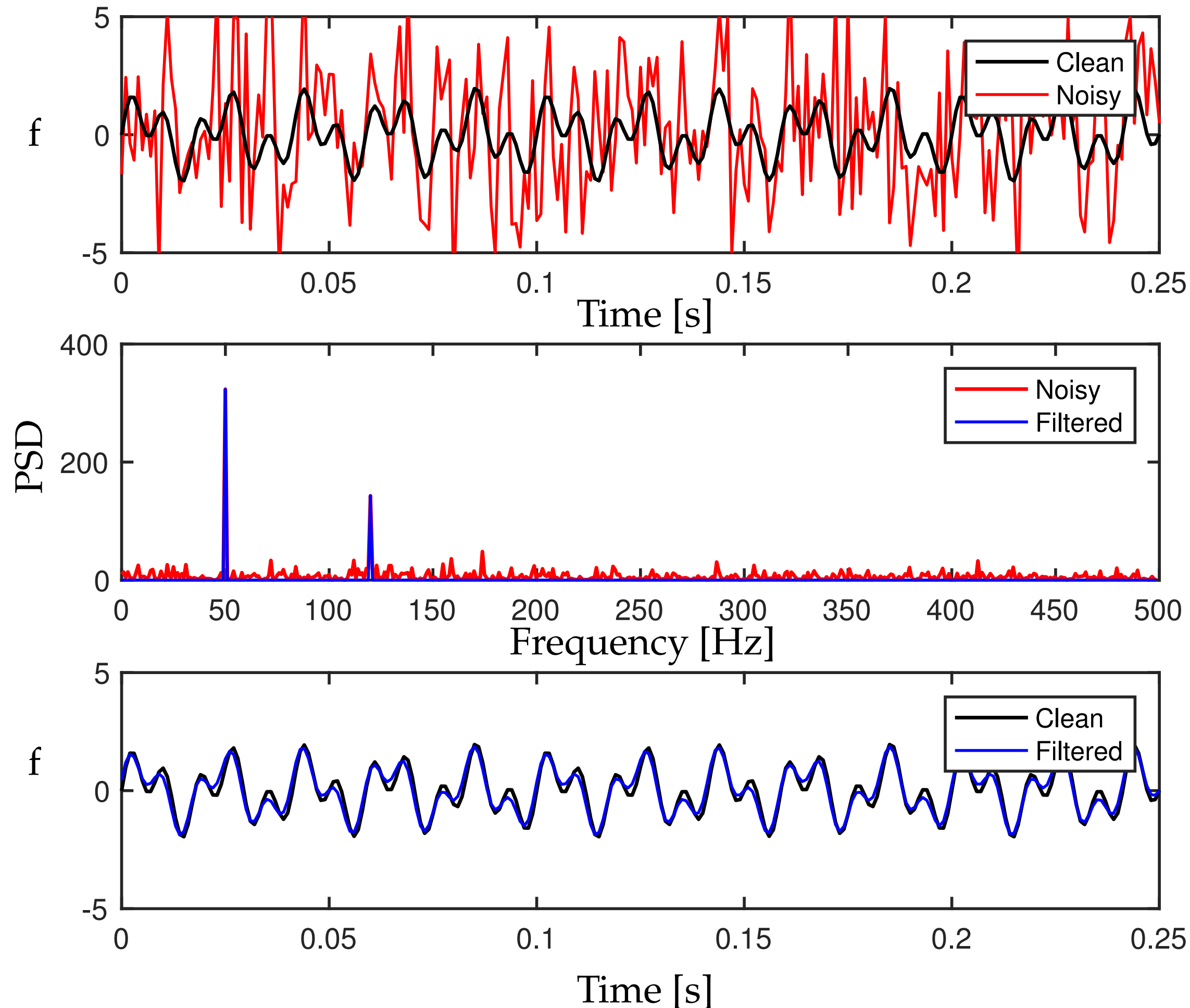
- f now is a N -dimensional vector; it can be represented by projecting to N mutually orthogonal vectors.
- DFT transforms a N -dimensional vector to another N -D vector.

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \dots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)^2} \end{bmatrix} \times \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-2} \\ f_{n-1} \end{bmatrix} = \begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \vdots \\ \hat{f}_{n-2} \\ \hat{f}_{n-1} \end{bmatrix}$$

Discrete Fourier Transform



Another Application: Signal Denoising



$$f(t) = \sin(2\pi f_1 t) + \sin(2\pi f_2 t) + \text{noise}$$

$$f_1 = 50 \quad f_2 = 120$$

$$\hat{f}(\omega) = DFT(f(t))$$

Remove low-magnitude
frequency components $\hat{f}'(\omega)$

$$f'(t) = IDFT(\hat{f}'(\omega))$$

2D DFT

2D Basis Functions

Real Bases

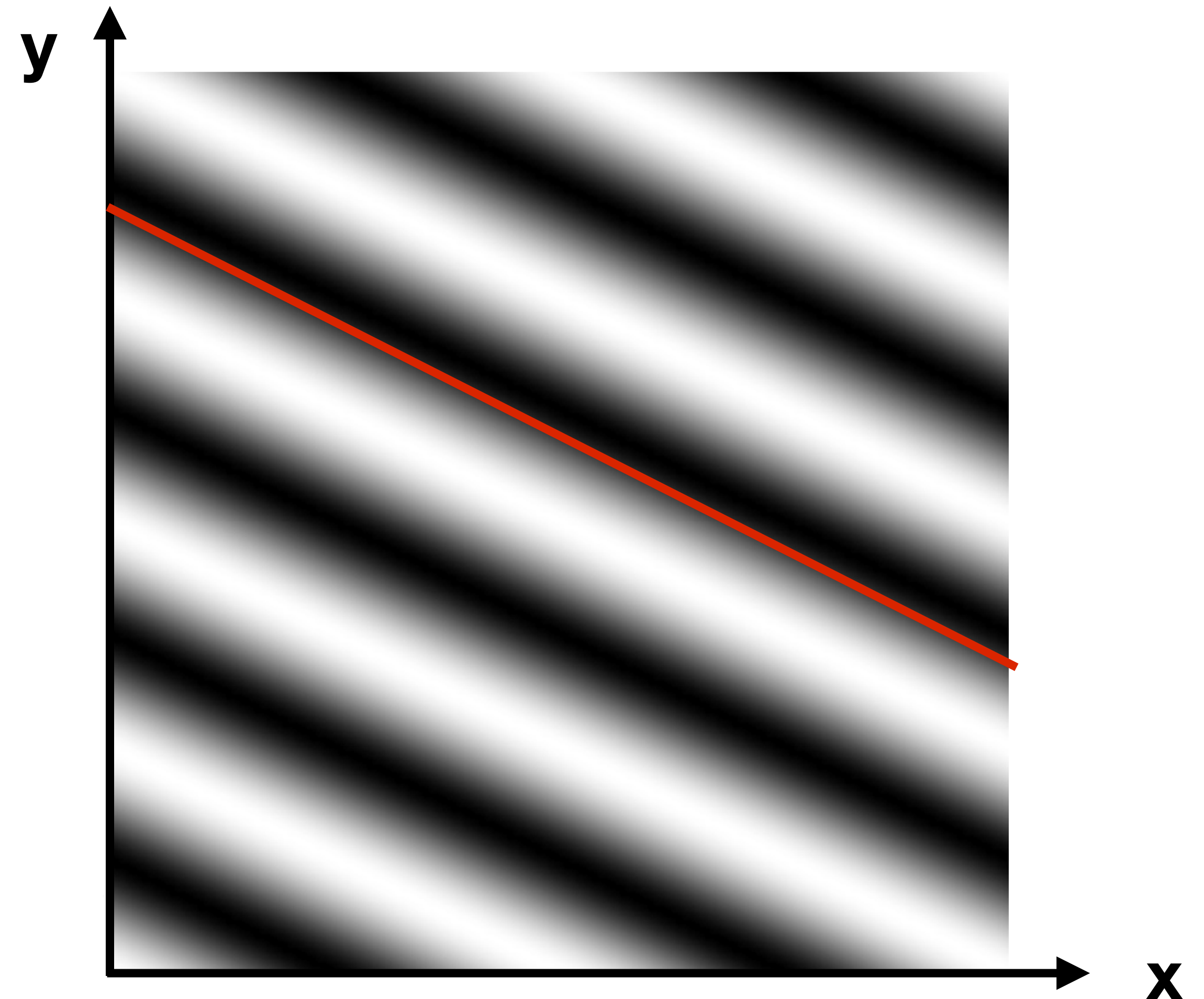
$$\left\{ \sin(ux + vy), \cos(ux + vy) \right\}, (u, v) \in \mathbb{Z}$$

Complex Bases

$$e^{i(ux+vy)}, (u, v) \in \mathbb{Z}$$

Function value is constant
where $x+2y$ is constant

"image" of $\cos(x + 2y)$



2D Discrete Fourier Transform

1D DFT

$$f_k = \frac{1}{n} \sum_{j=0}^{n-1} \hat{f}_k e^{\frac{i2\pi jk}{n}}$$

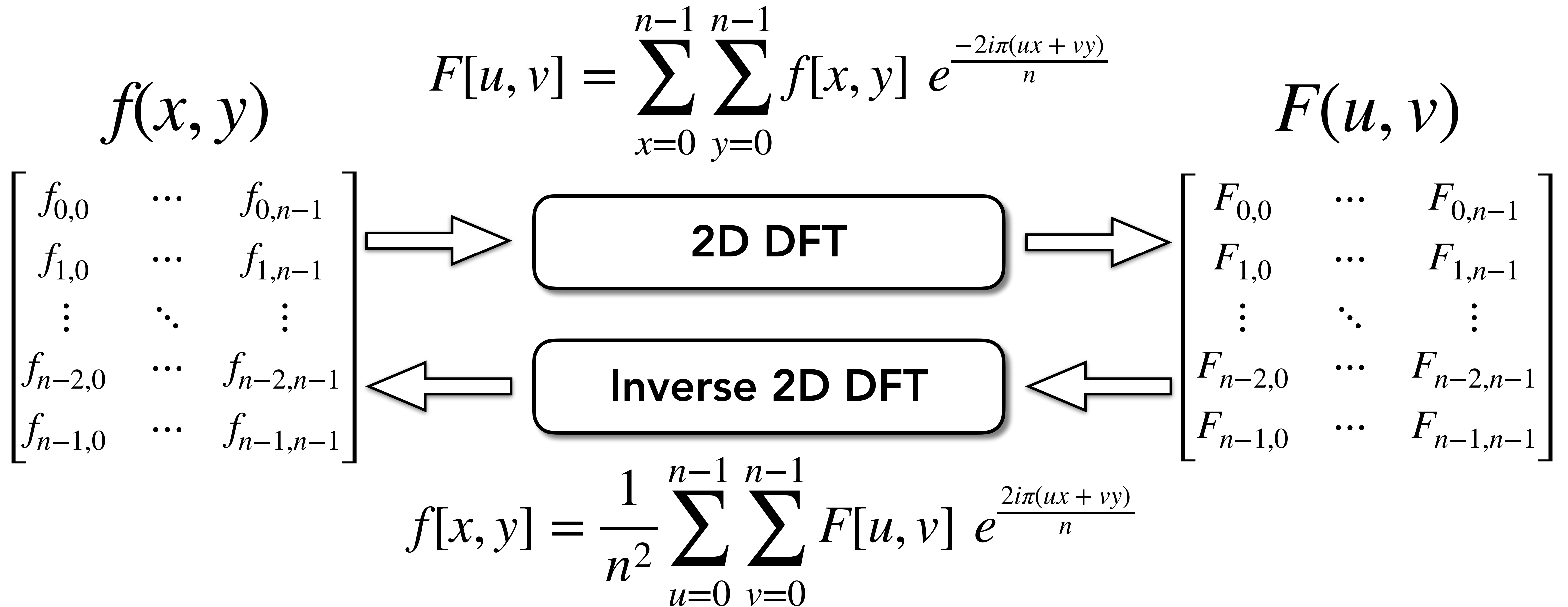
$$\hat{f}_k = \sum_{j=0}^{n-1} f_j e^{\frac{-i2\pi jk}{n}}$$

2D DFT

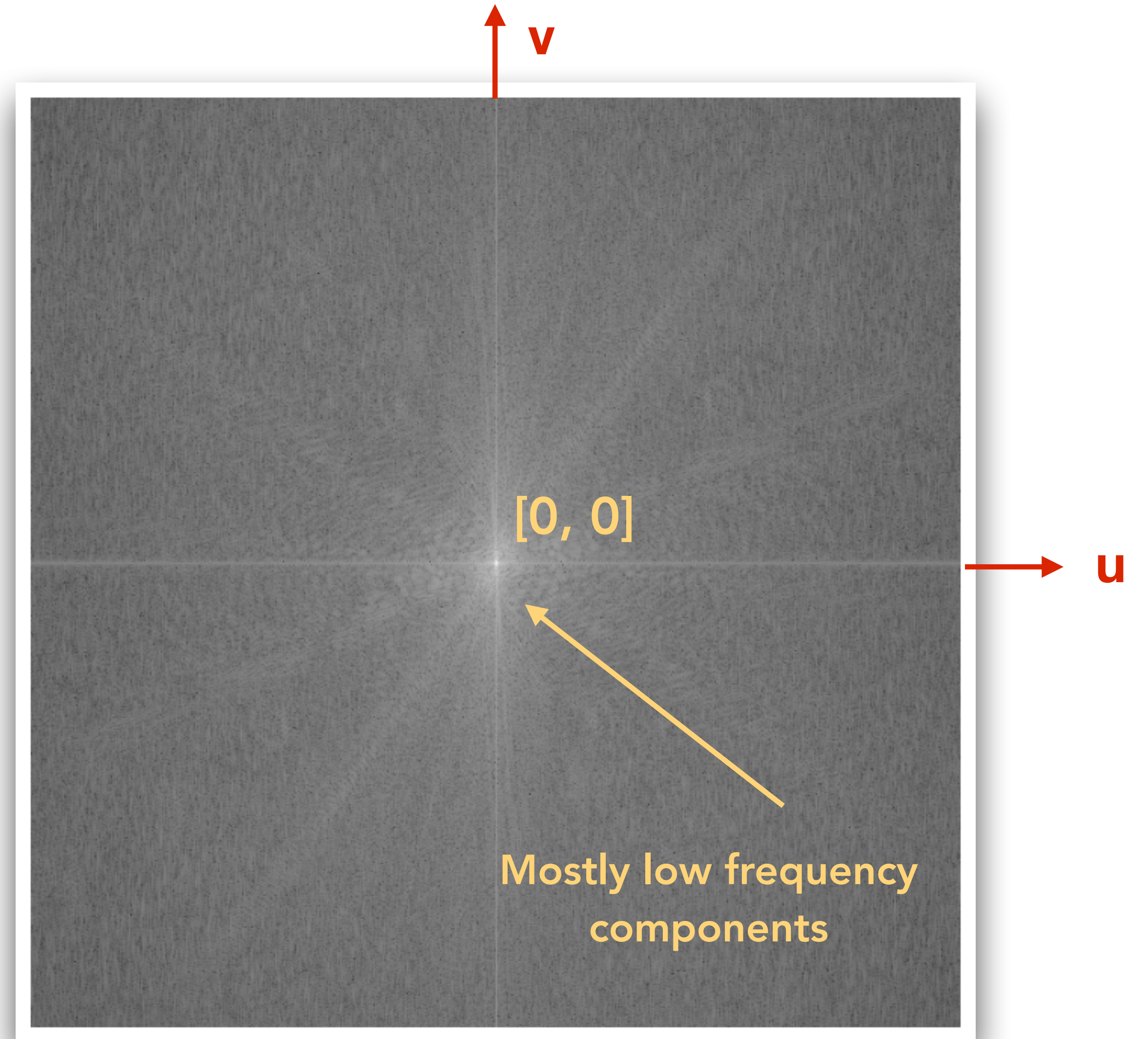
$$f[x, y] = \frac{1}{n^2} \sum_{u=0}^{n-1} \sum_{v=0}^{n-1} F[u, v] e^{\frac{2i\pi(ux + vy)}{n}}$$

$$F[u, v] = \sum_{x=0}^{n-1} \sum_{y=0}^{n-1} f[x, y] e^{\frac{-2i\pi(ux + vy)}{n}}$$

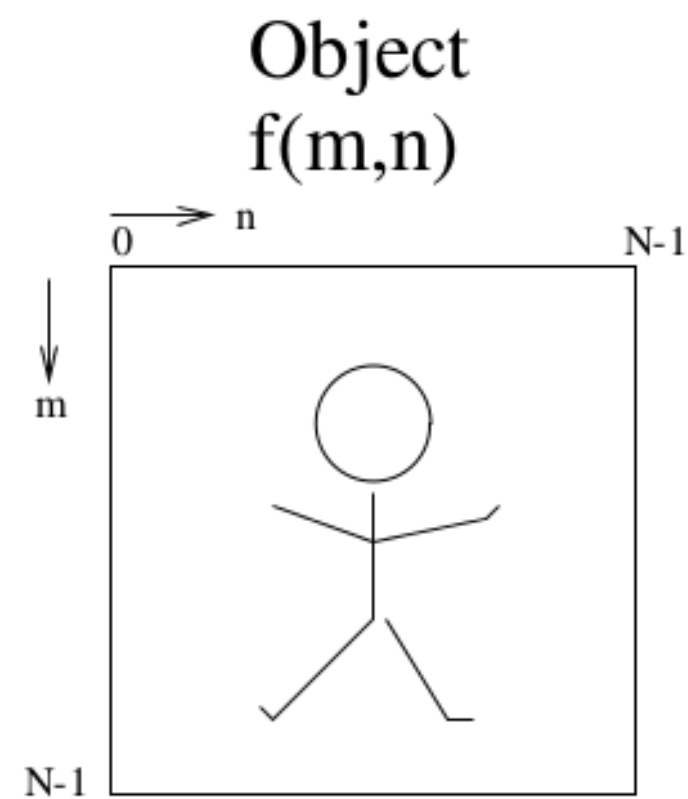
2D DFT



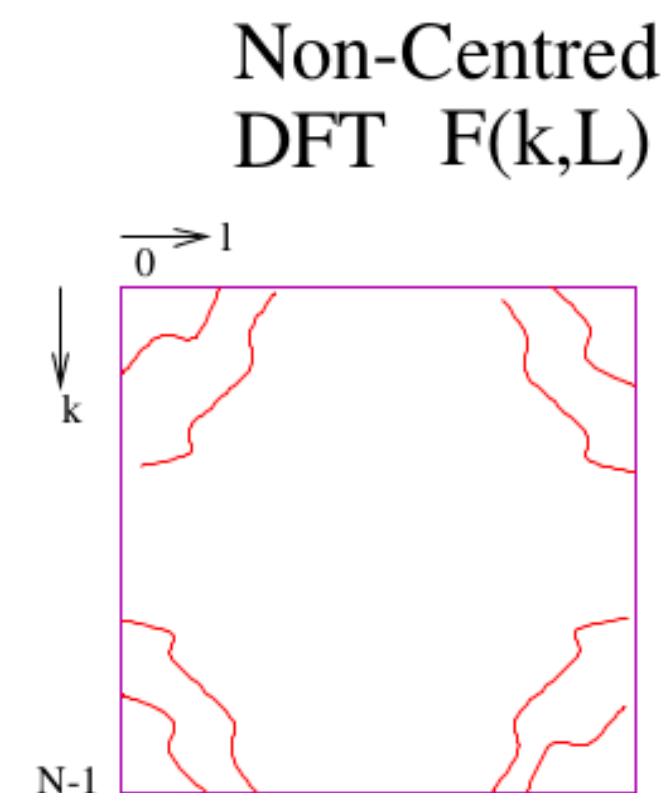
2D DFT Coefficients of a Typical Image



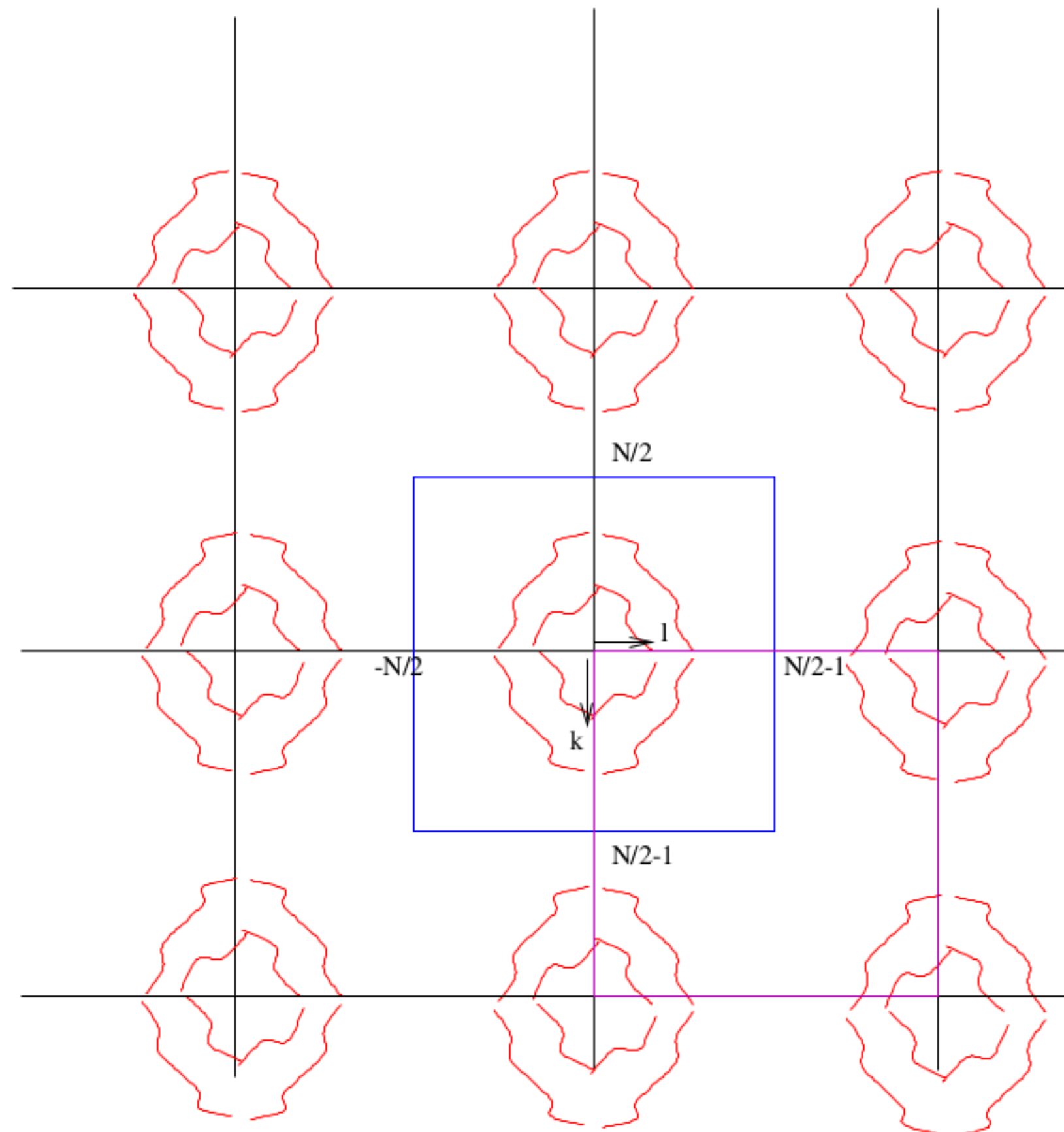
DFT Coefficients Are Periodic



DFT



Extended DFT Calculated at all k,l

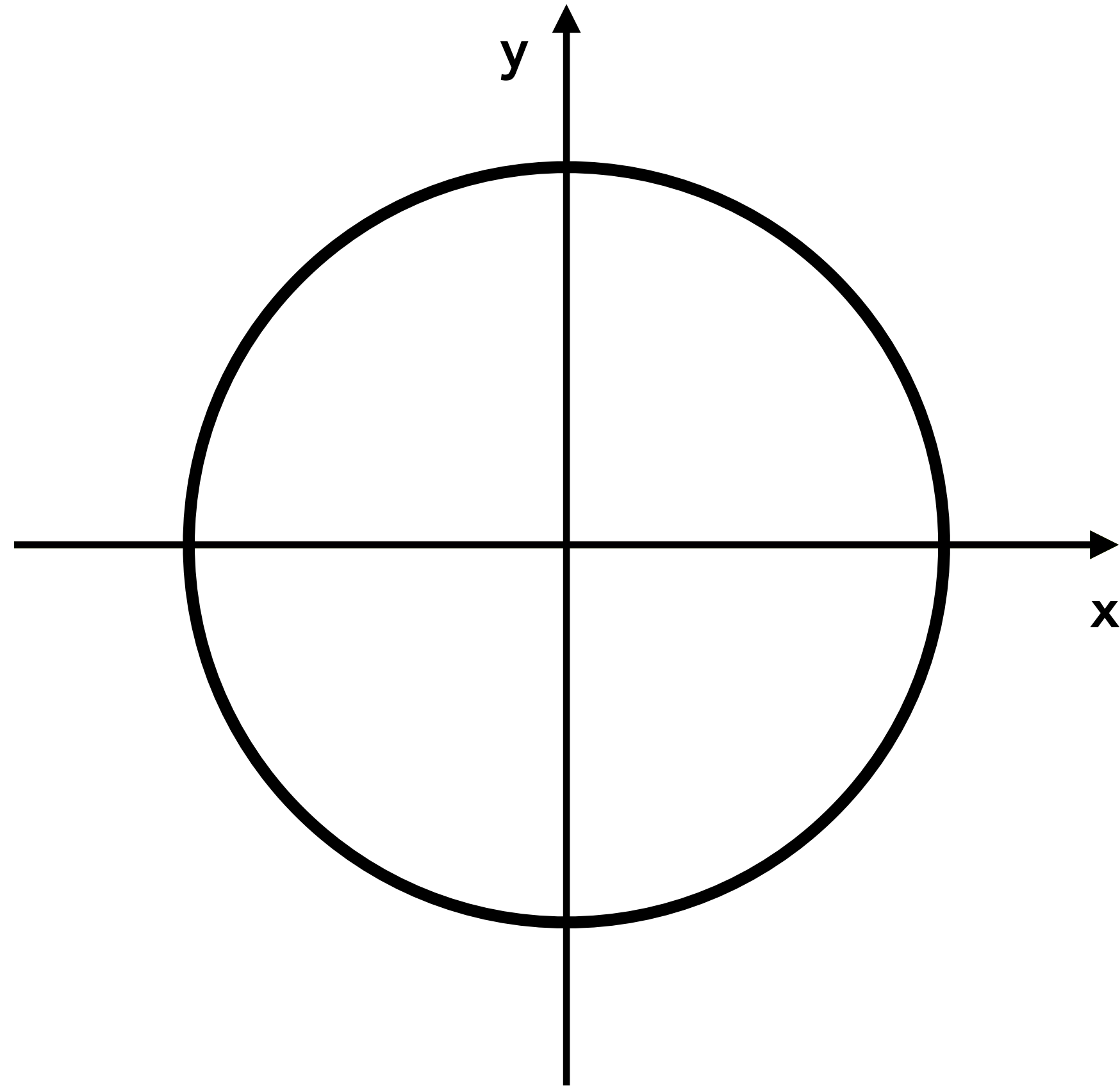


$$F[u, v] = \sum_{x=0}^{n-1} \sum_{y=0}^{n-1} f[x, y] e^{\frac{-2i\pi(ux + vy)}{n}}$$

$$F[k_1, k_2] = F[n + k_1, n + k_2]$$

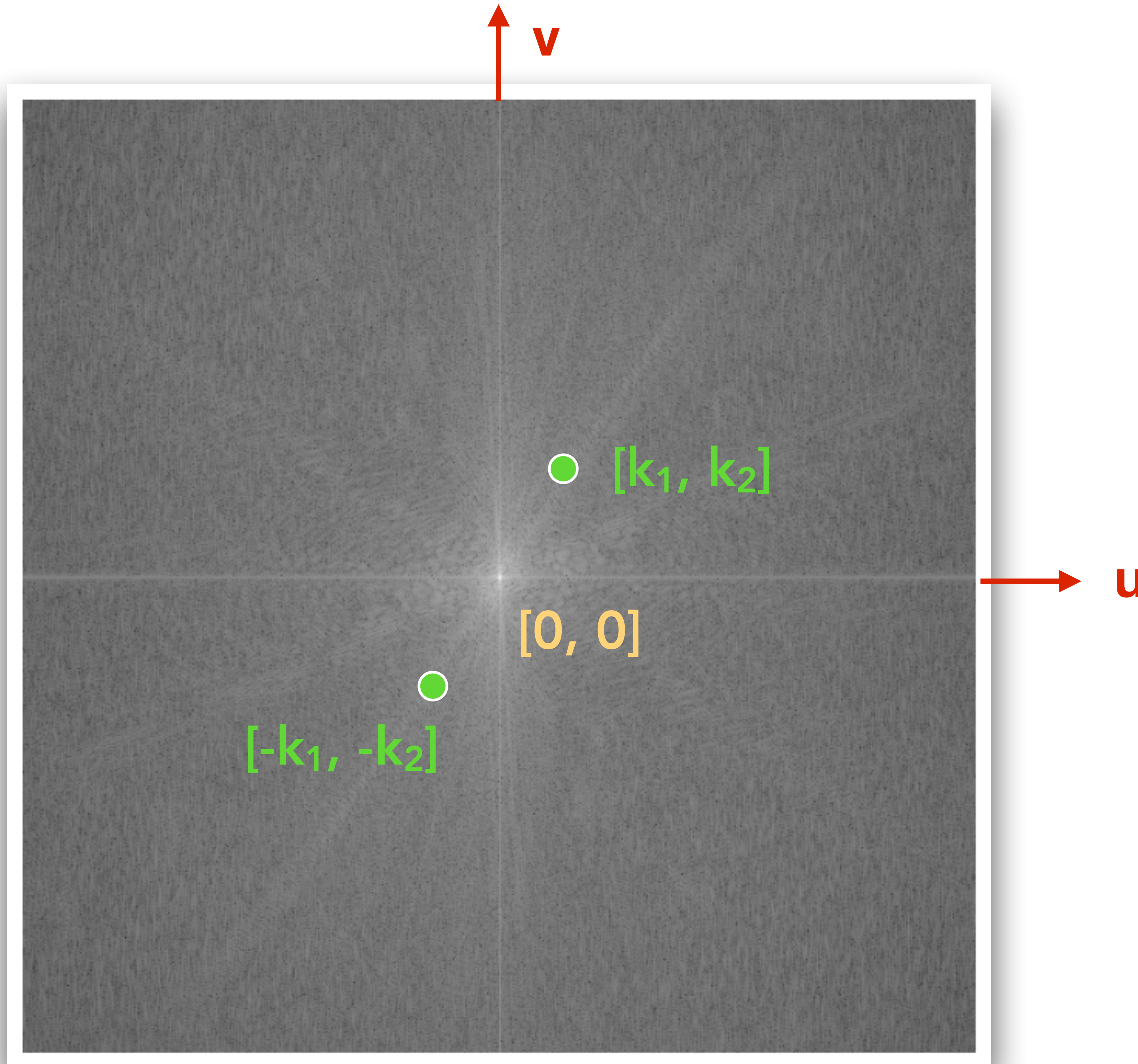
$$F[k_1, k_2] = \bar{F}[n - k_1, n - k_2]$$

What are "Negative" Frequencies?



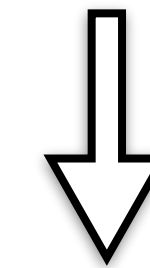
- In this example:
 - Counterclockwise rotation has an angular frequency of $1/8$ Hz (cycles/second)
 - Clockwise rotation has a frequency of $7/8$ Hz
 - If we sample the position of the wheel every 1s, both rotations have the same observable effect
 - If clockwise rotation has a positive frequency, counterclockwise rotation has a **negative** freq.
- Aliasing: when we are sampling not frequently enough, high freq. info. masquerades as low freq. info.

DFT Coefficients are Symmetric in Magnitude



$$F[k_1, k_2] = F[n + k_1, n + k_2]$$

$$F[k_1, k_2] = \bar{F}[n - k_1, n - k_2]$$



$$F[-k_1, -k_2] = F[n - k_1, n - k_2]$$

$$F[-k_1, -k_2] = \bar{F}[k_1, k_2]$$

$$|F[-k_1, -k_2]| = |F[k_1, k_2]|$$

Fourier Transform

What About Non-Periodic Functions?

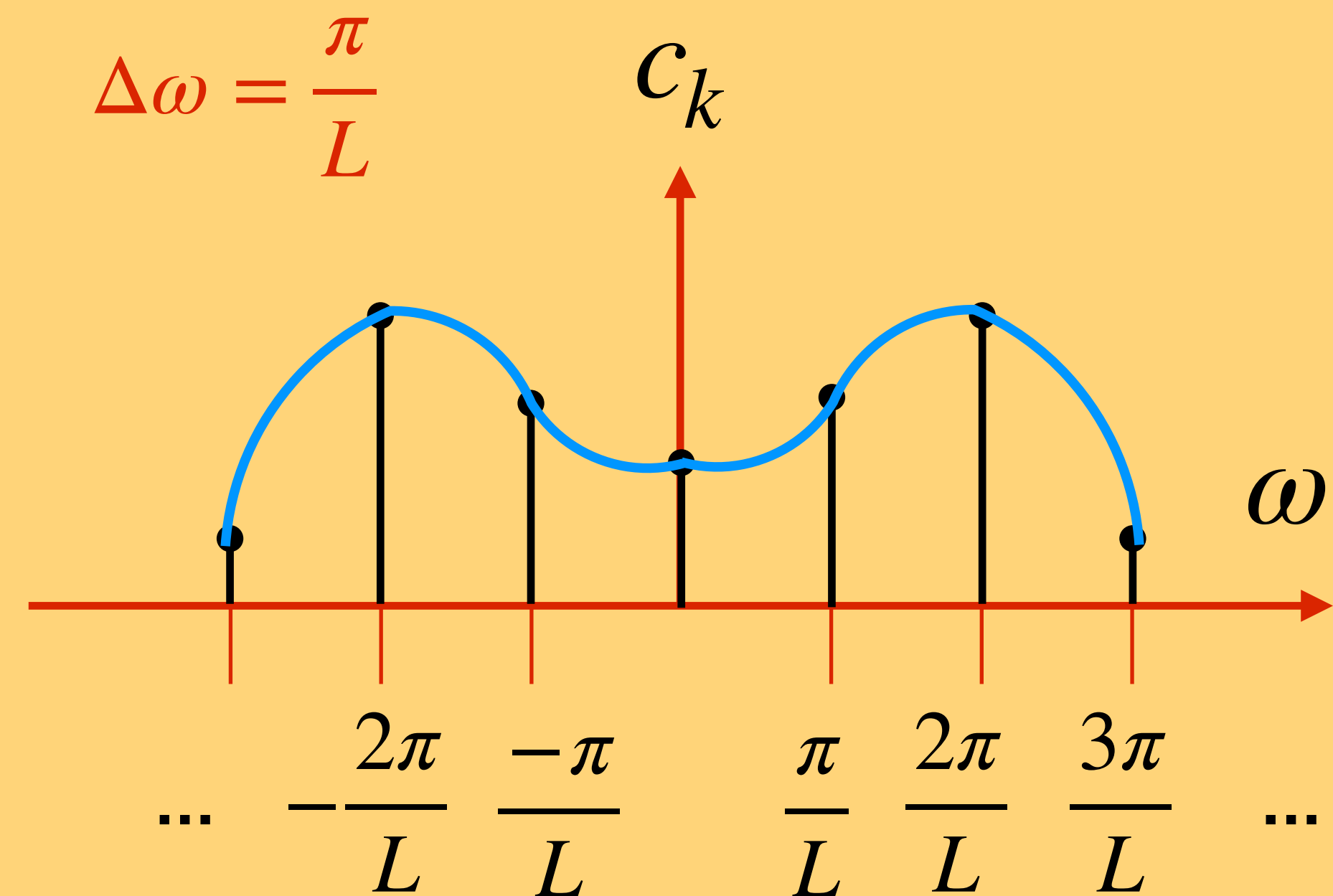
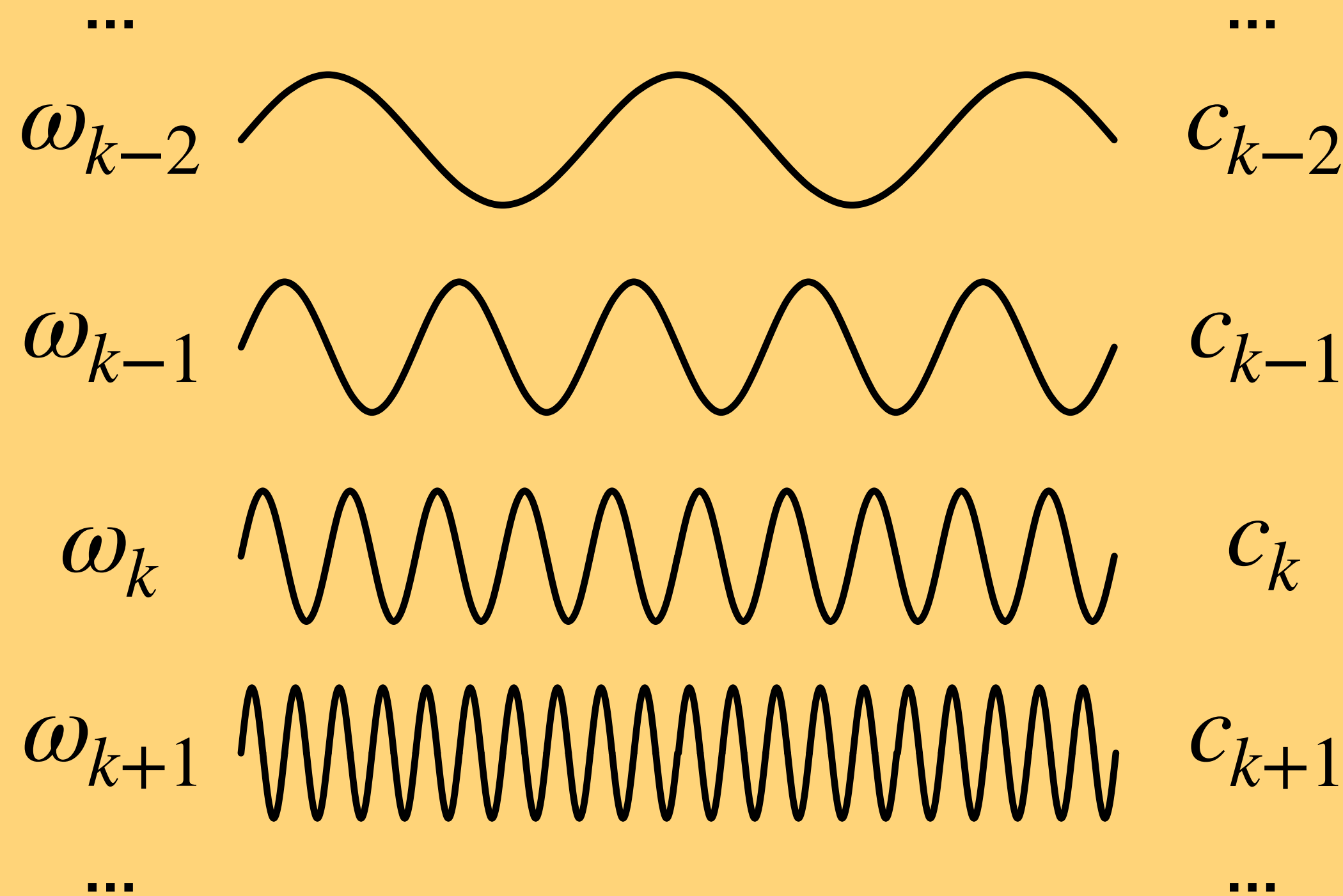
- Fourier series is concerned with **L-periodic** functions.
- What about a general, potentially non-periodic function? Does the Fourier theory still apply to it?
- Idea: let **L approach infinity** and see how the Fourier series change!

As L Approaches Infinity

$$\omega_k = \frac{k\pi}{L} \quad f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i\omega_k x} \quad c_k = \frac{1}{2L} \int_{-L}^L f(x) e^{-i\omega_k x} dx$$

Frequency

Coeff.



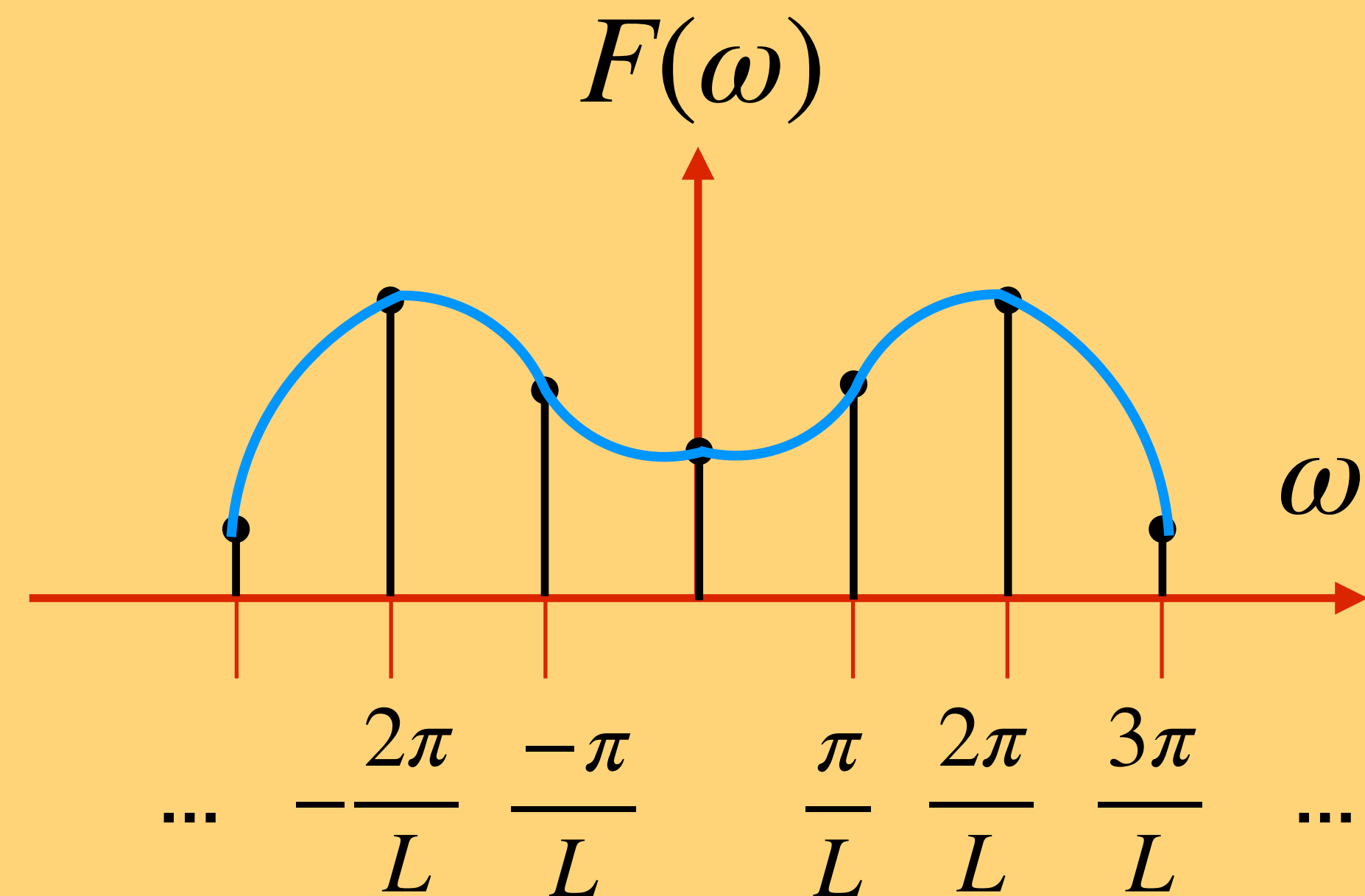
As L Approaches Infinity

$$\omega_k = \frac{k\pi}{L}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i\omega_k x}$$

$$c_k = \frac{1}{2L} \int_{-L}^L f(x) e^{-i\omega_k x} dx$$

- As L approaches ∞ , the infinite sequence of Fourier coefficients becomes a continuous function $F(\omega)$.

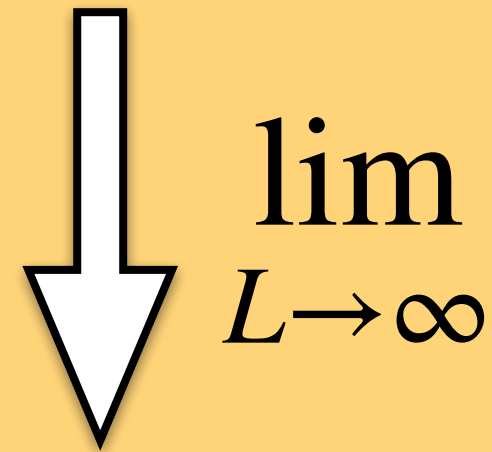


As L Approaches Infinity

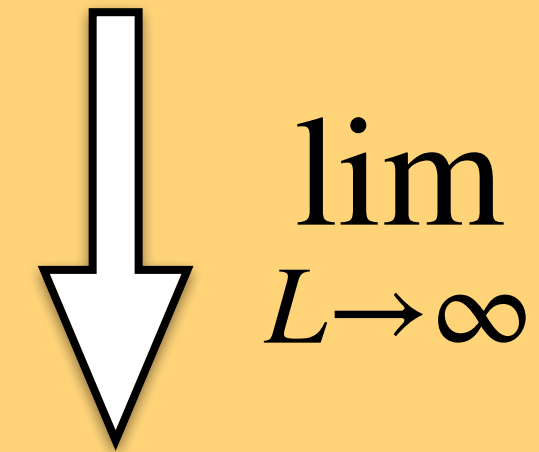
$$\omega_k = \frac{k\pi}{L}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i\omega_k x}$$

$$c_k = \frac{1}{2L} \int_{-L}^L f(x) e^{-i\omega_k x} dx$$

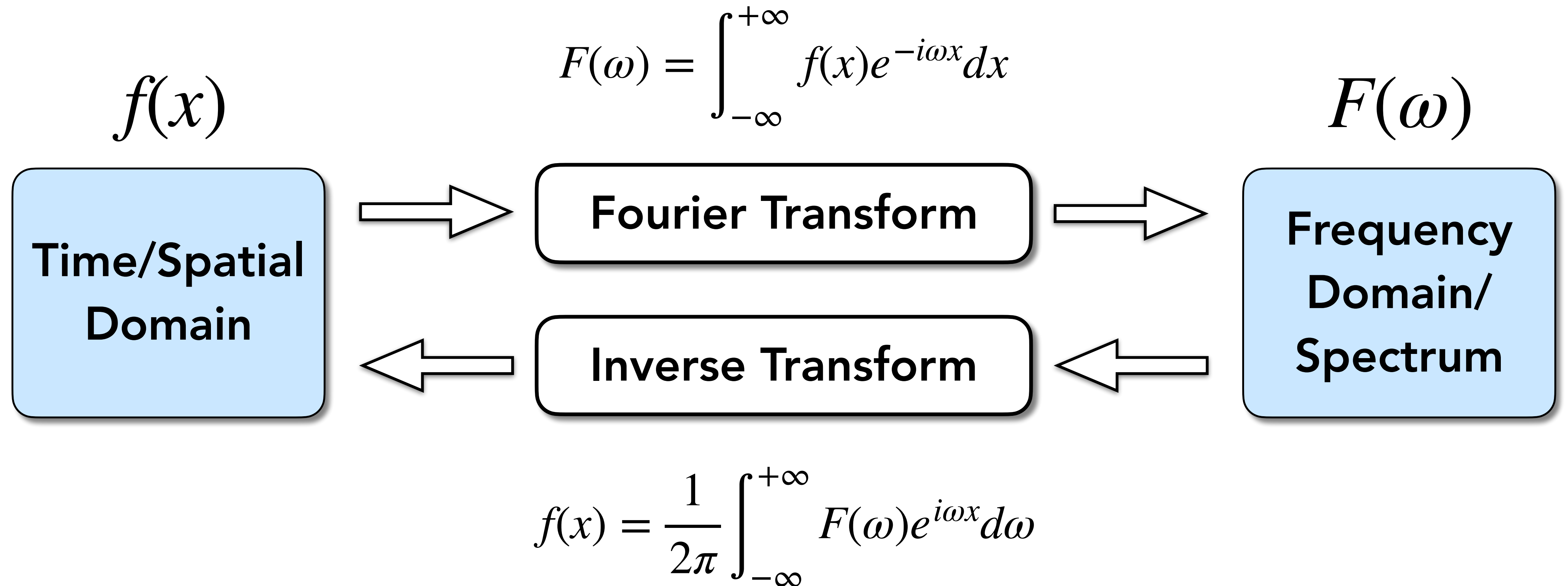


$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega x} d\omega$$



$$F(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx$$

Fourier Transform



Key Things to Take Away

- Signals
 - Audio waves are discrete 1D signals (in the time domain)
 - Images are discrete 2D signals (in the spatial domain)
- Any time/spatial domain signals can be converted to frequency domain
 - Frequency domain representation of signal is called **spectrum**
- Fourier series applies to continuous L-period signals
- Discrete Fourier Transform applies to discrete L-period signals
- Fourier Transform applies to non-periodic continuous signals

Key Things to Take Away

- How to tell frequency from images?
 - “Busy” areas or sudden changes (e.g., edge) correspond to high frequency components
 - Smooth areas are generally of lower frequency
- Removing low/high frequency components from the spectrum corresponds to sharpening/blurring images in spatial domain