Lecture 2: Geometric Transformation

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CSC 259/459, Fall 2024
Computer Imaging & Graphics

Logistics

Assignment 0 is up and is due Sept. 4 (Wednesday) 11:30 AM.

Course schedule: https://cs.rochester.edu/courses/259/fall2024/schedule.html. You will find reading assignments and slides.

- Some readings are on Box, so you will need to log in using your UR credentials.

A0 will be submitted through Blackboard.

Start thinking and talking to me about your final project idea.

The Roadmap

Theoretical Preliminaries

Human Visual Systems

Color in Nature, Arts, & Tech
(a.k.a., the birth, life, and death of light)

Digital Camera Imaging

Modeling and Rendering

Applications

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Geometric Transformations

Fourier Series & Transforms

Sampling & Reconstruction

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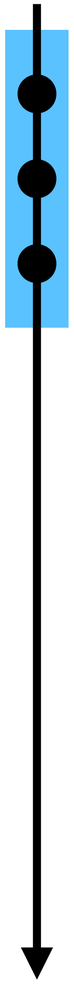
Applications



Geometric Transformations

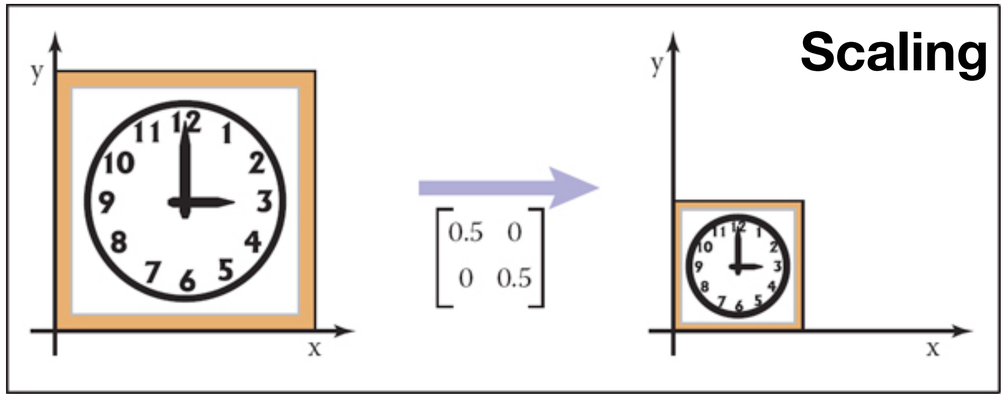
Fourier Series & Transforms

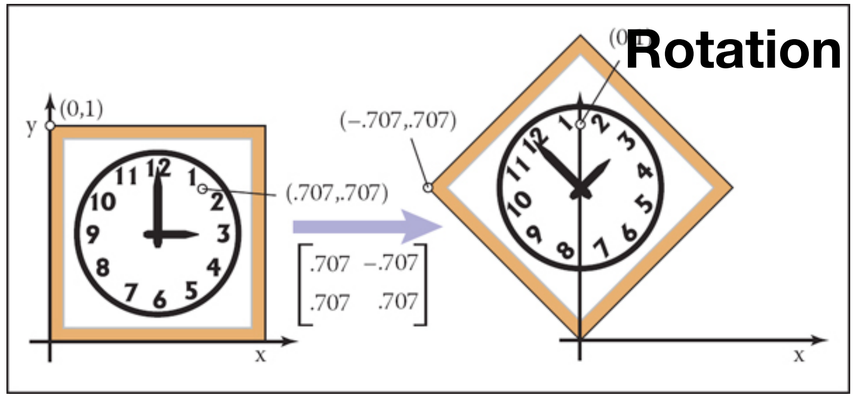
Sampling & Reconstruction

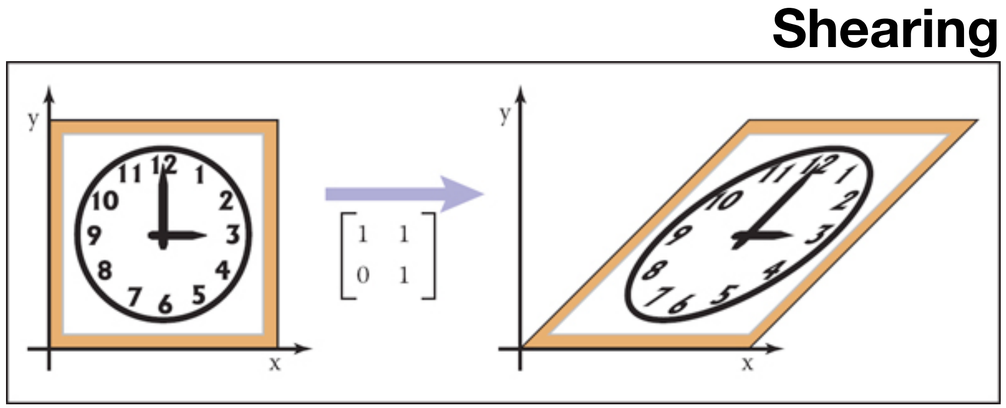
Assumes basic understanding of linear algebra. A nice review of linear algebra is posted on the schedule page too.

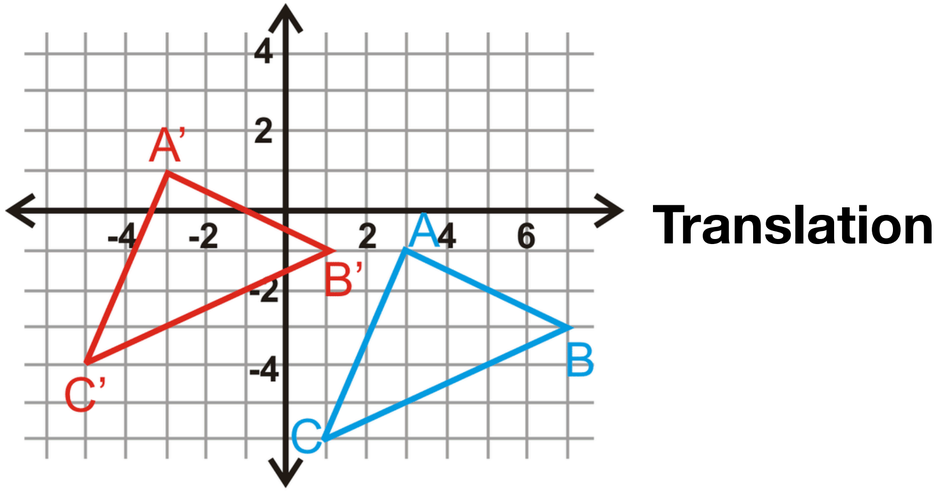
Building Intuitions

Some Examples of Geometric Transformations (2D)



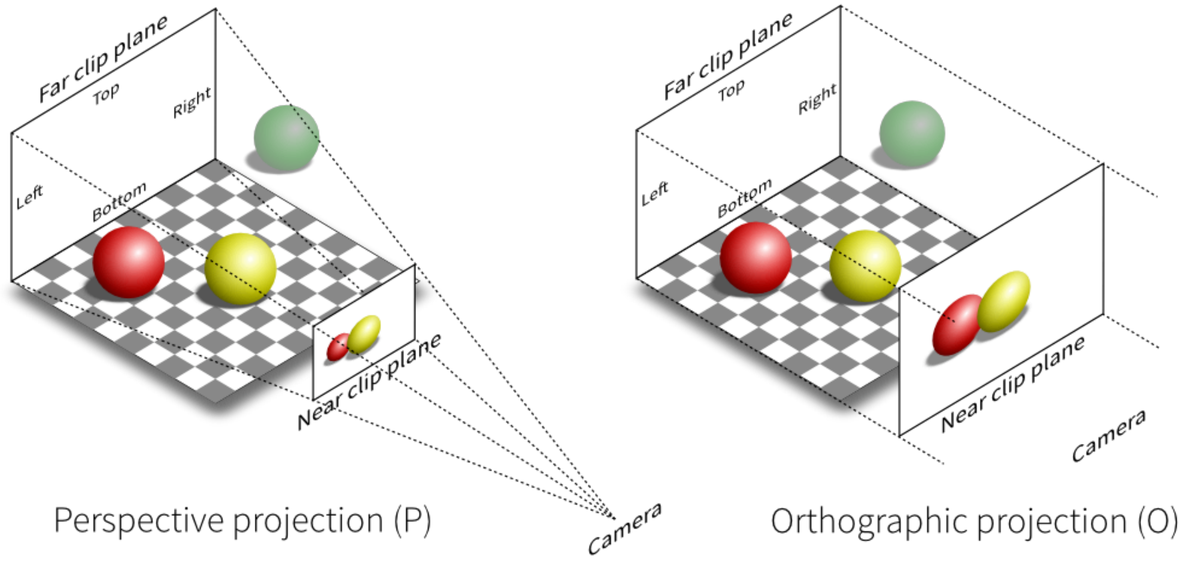


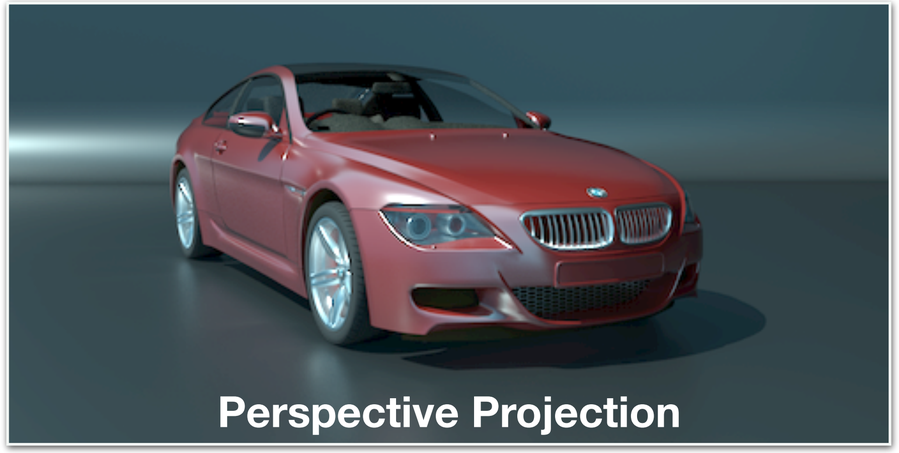


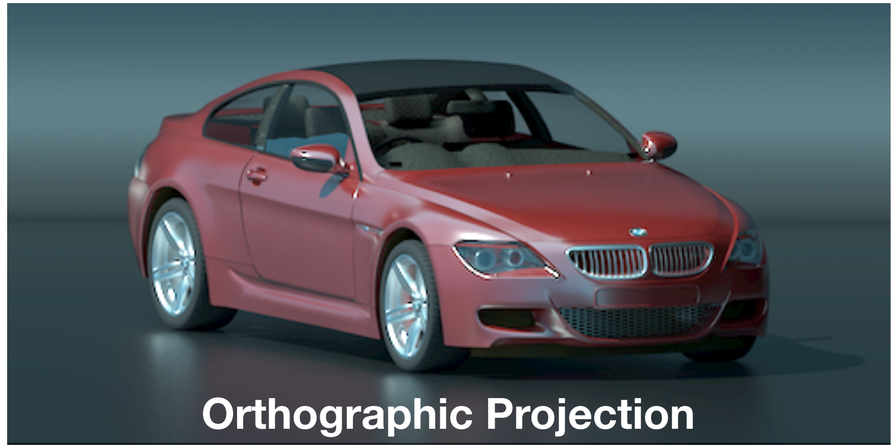


https://www.ck12.org/geometry/translations/lesson/Geometric-Translations-BSC-GEOM/

3D to 2D Transformations (e.g., Camera)







http://www.pbr-book.org/3ed-2018/Camera_Models/Environment_Camera.html

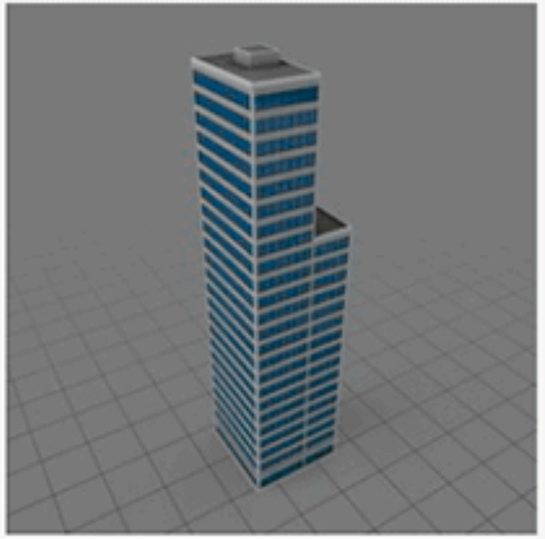
Perspective Projection

https://www.adorama.com/alc/outdoor-architecture-photography-tip

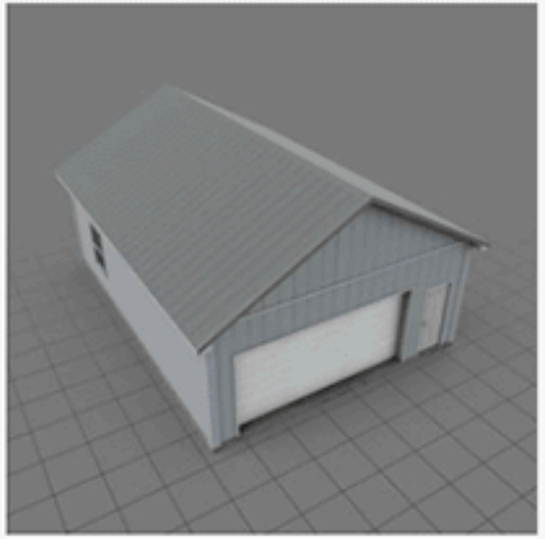
Orthographic Projection

https://gamedev.stackexchange.com/questions/43281/how-to-add-isometric-rts-alike-perspective-and-scolling-in-unity


ROCKET
BRUSH
STUDIO

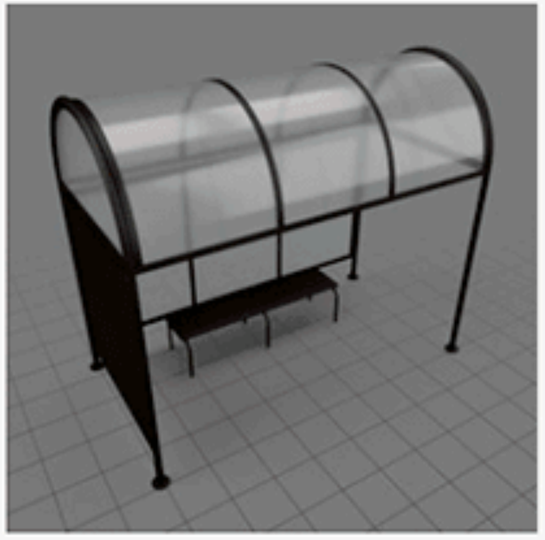


Office building **model**

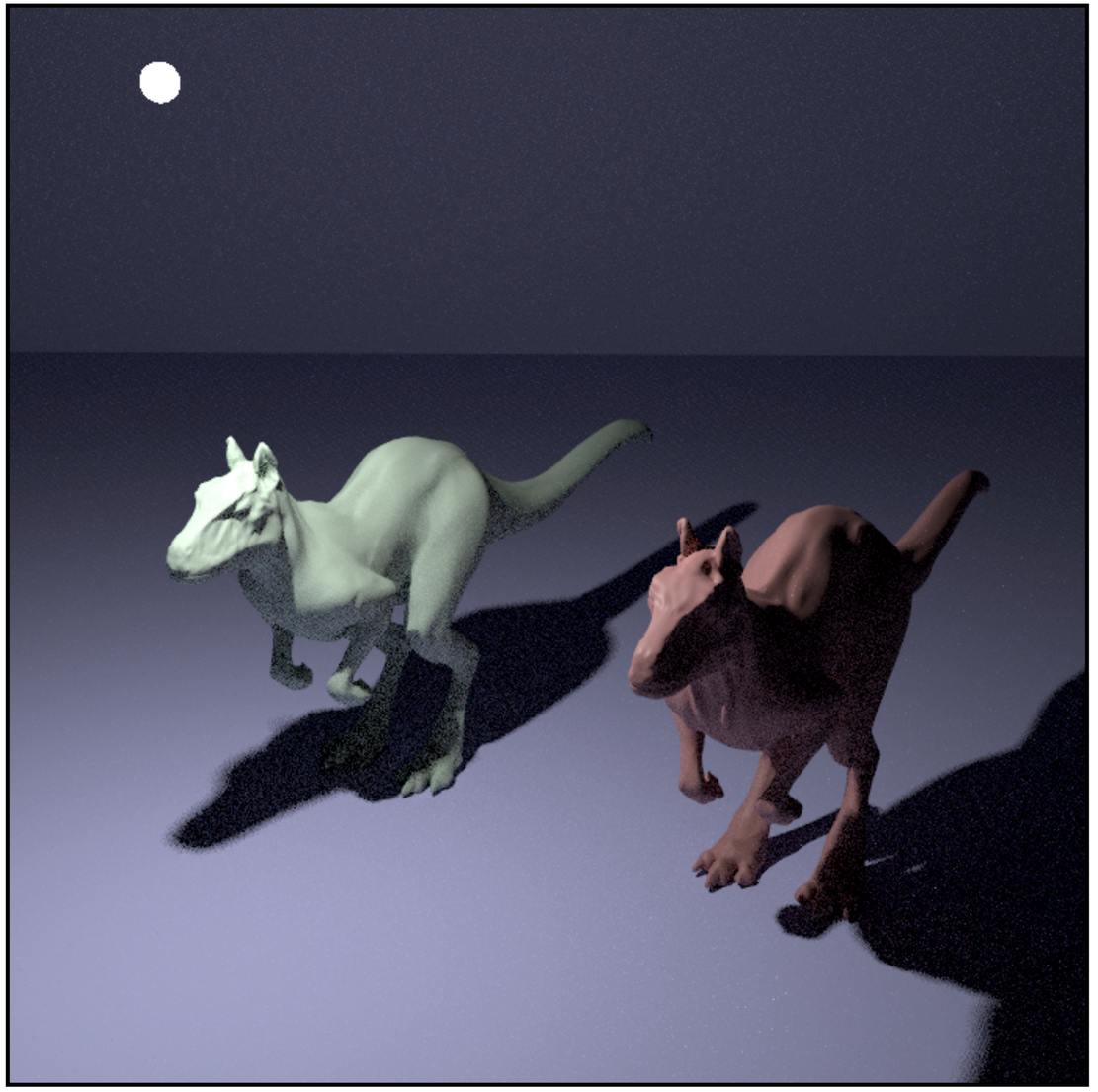


Model Pole barn garage

Model Male statues

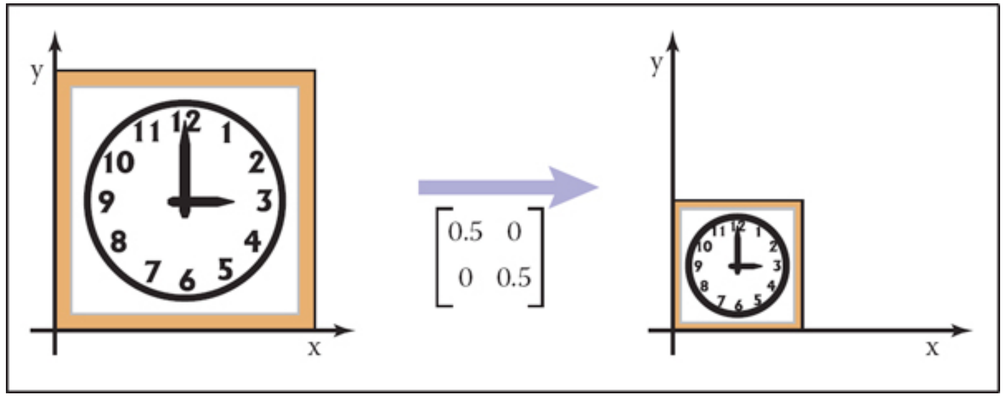


Bus stop **model**



https://fixthephoto.com/free-3d-models.html

Building the Intuition



As if a force is applied to **all** the points in the input model

Key: all the point in the input are transformed in some **systematic** way

A point P [x, y, z]. Think of it as a 1x3 matrix

Transformation: change P [x, y, z] to P’ [x’, y’, z’]

Point Transformations

Geometric Transformation: What Is It?

We focus on transformations that can be expressed as matrix multiplication

- Many important transformations that we care about can be expressed as matrix multiplication

Different transformations require different matrices.

$$[x, y, z] \quad x \quad \begin{bmatrix} T_{00}, T_{01}, T_{02} \\ T_{10}, T_{11}, T_{12} \\ T_{20}, T_{21}, T_{22} \end{bmatrix} \quad = \quad [x', y', z']$$

Identity Transformation

Point Transformation using Matrix Multiplication

What should T be like if we want to keep x the same before and after the transformation — regardless of where P [x, y, z] is.

$$x' = xT_{00} + yT_{10} + zT_{20} = x, \text{ for } \forall\ x, y, z$$

$$1 \qquad 0 \qquad 0$$

$$[x, y, z] \quad x \quad \begin{bmatrix} 1 & T_{01}, T_{02} \\ 0 & T_{11}, T_{12} \\ 0 & T_{21}, T_{22} \end{bmatrix} \quad = \quad [x', y', z']$$

Point Transformation using Matrix Multiplication

What should T be like if we want to keep y the same before and after the transformation — regardless of where P [x, y, z] is.

$$y' = xT_{01} + yT_{11} + zT_{21} = y, \textit{ for } \forall\ x, y, z$$

$$0 \qquad 1 \qquad 0$$

$$[x, y, z] \quad x \quad \begin{bmatrix} T_{00}, & 0 & T_{02} \\ T_{10}, & 1 & T_{12} \\ T_{20}, & 0 & T_{22} \end{bmatrix} \quad = \quad [x', y', z']$$

Point Transformation using Matrix Multiplication

What should T be like if we want to keep z the same before and after the transformation — regardless of where P [x, y, z] is.

$$z' = xT_{02} + yT_{12} + zT_{22} = z, \text{ for } \forall\ x, y, z$$

$$0 \quad 0 \quad 1$$

$$[x, y, z] \quad x \quad \begin{bmatrix} T_{00}, & T_{01}, & 0 \\ T_{10}, & T_{11}, & 0 \\ T_{20}, & T_{21}, & 1 \end{bmatrix} \quad = \quad [x', y', z']$$

Identity Matrix

What should T be like if we want to keep a point unchanged before and after the transformation — regardless of where P [x, y, z] is?

That matrix is called the identity matrix

$$[x, y, z] \quad x \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad = \quad [x', y', z']$$

Scaling

Scaling

Changing from P [x, y, z] to P' [$S_0 \cdot x$, $S_1 \cdot y$, $S_2 \cdot z$]

The "scaling factor": [S_0, S_1, S_2]

How should the transformation matrix look like?

$$x' = xT_{00} + yT_{10} + zT_{20} = S_0x$$

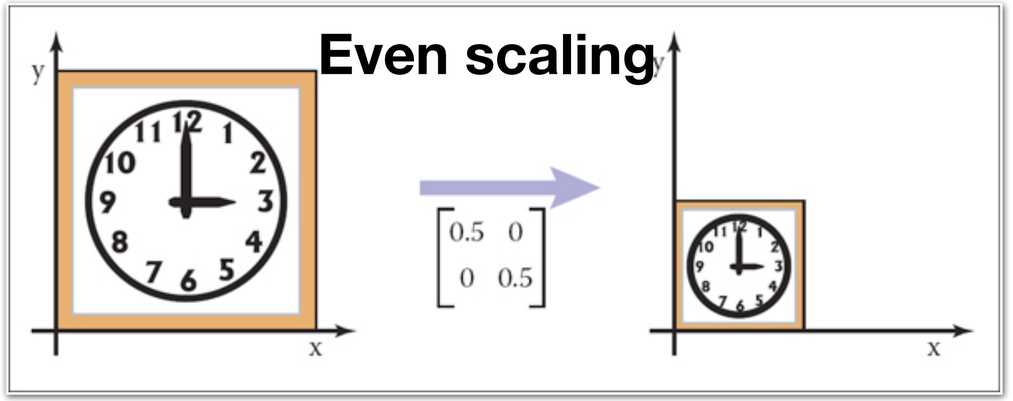
Scaling matrix is a diagonal matrix

$$[x, y, z] \quad x \quad \begin{bmatrix} S_0 & 0 & 0 \\ 0 & S_1 & 0 \\ 0 & 0 & S_2 \end{bmatrix} = [x', y', z']$$

Scaling

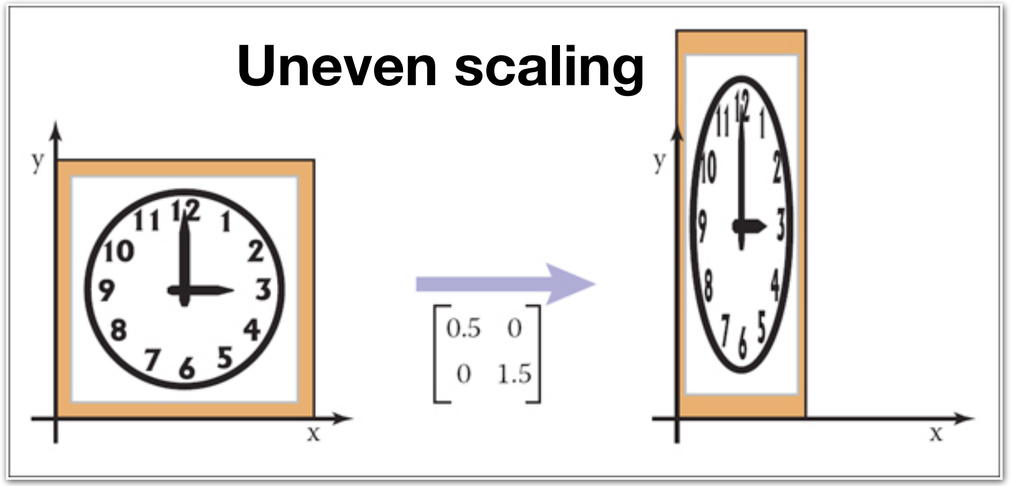
$$[x, y] \times \begin{bmatrix} 0.5, 0 \\ 0, 0.5 \end{bmatrix} = [x', y']$$

$$x' = 0.5x$$
$$y' = 0.5y$$



$$[x, y] \times \begin{bmatrix} 0.5, 0 \\ 0, 1.5 \end{bmatrix} = [x', y']$$

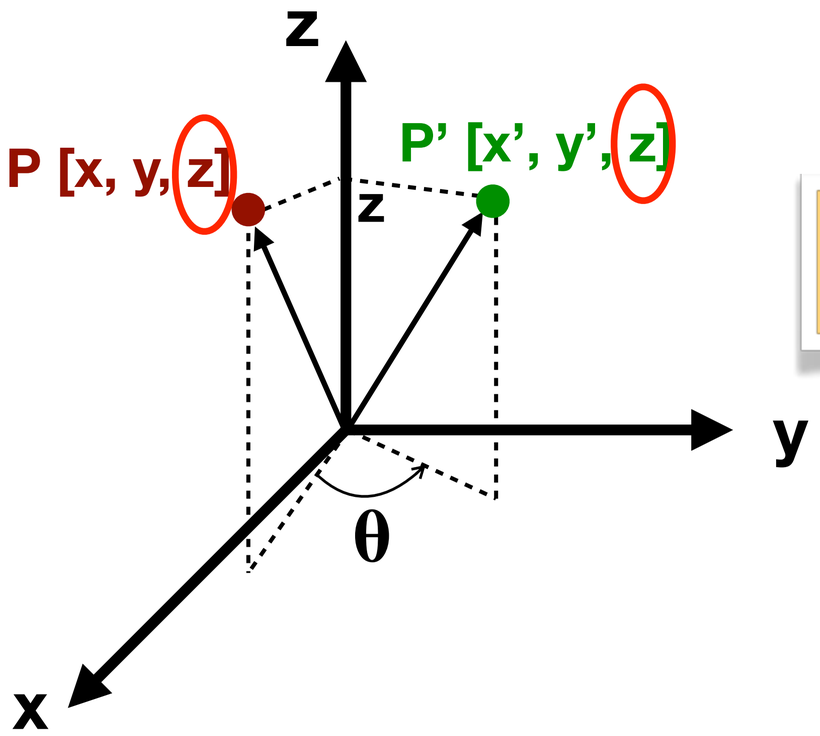
$$x' = 0.5x$$
$$y' = 1.5y$$



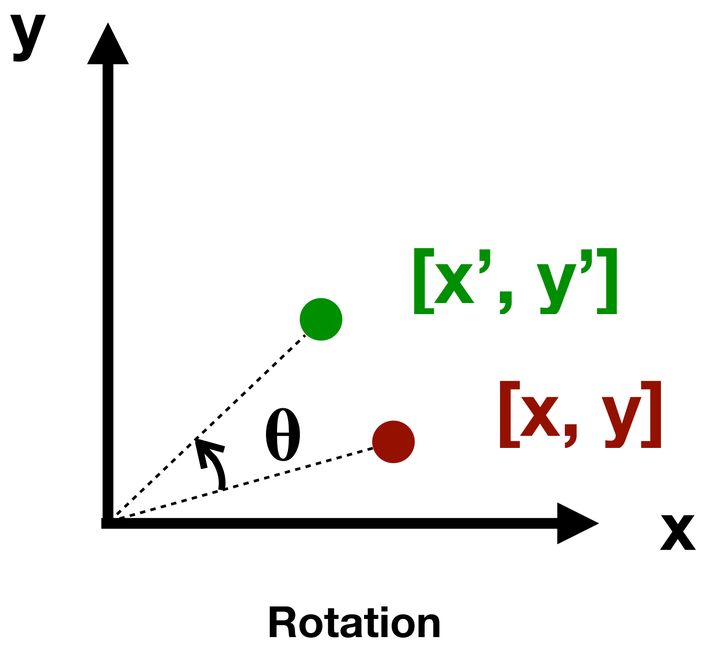
Rotation

Rotation

What should the transformation matrix be to rotate P around the z-axis by θ, regardless what P is?

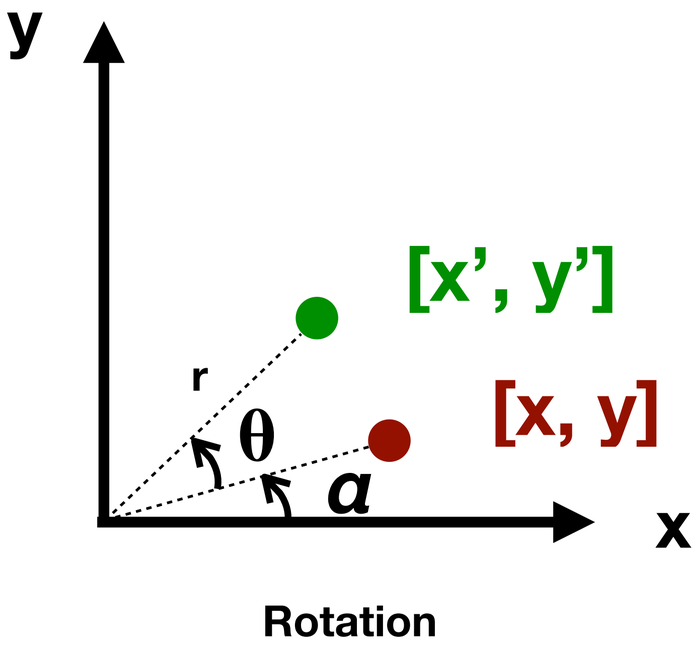


Keep z the same, rotate within the x-y plane



Rotation

Rotation



Rotation

$\sin \alpha = y / r$

$\cos \alpha = x / r$

$\sin(\alpha + \theta) = y' / r = \sin \alpha * \cos \theta + \cos \alpha * \sin \theta$

$= y / r * \cos \theta + x / r * \sin \theta$

$y' / r = y / r * \cos \theta + x / r * \sin \theta$

$y' = y * \cos \theta + x * \sin \theta$

$\cos(\alpha + \theta) = x' / r = \cos \alpha * \cos \theta - \sin \alpha * \sin \theta$

$= x / r * \cos \theta - y / r * \sin \theta$

$x' / r = x / r * \cos \theta - y / r * \sin \theta$

$x' = x * \cos \theta - y * \sin \theta$

Rotation Matrix (Around Z-axis)

$x' = x \cos\theta - y \sin\theta = xT_{00} + yT_{10} + zT_{20}$, *for* $\forall\ x, y, z$

$y' = x \sin\theta + y \cos\theta = xT_{01} + yT_{11} + zT_{21}$, *for* $\forall\ x, y, z$

$z' = z = xT_{02} + yT_{12} + zT_{22}$, *for* $\forall\ x, y, z$

$$[x, y, z] \quad x \quad \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = [x', y', z']$$

z indeed doesn't change!

Rotation Matrix

About X

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

About Y

$$\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

About Z

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Derive the rest in your homework.

Unitary Matrix

$Q^T = Q^{-1}$ $Q\,Q^T = I$

Rotation matrix is a **unitary matrix.**

- The length (norm) of each row is 1.
- The rows are orthogonal vectors to each other.

Transpose is the same as inversion.

Orthogonal vectors:

- $\mathbf{v1}$ $[x_1, y_1, z_1]$ and $\mathbf{v2}$ $[x_2, y_2, z_2]$ are orthogonal if $\mathbf{v1}\cdot\mathbf{v2} = x_1x_2 + y_1y_2 + z_1z_2 = 0$.
- $\mathbf{v1}\cdot\mathbf{v2}$ is called the **dot (inner) product**.
- Orthonormal vectors are orthogonal, unit vectors.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \leftarrow \mathbf{v1} \\ \leftarrow \mathbf{v2} \\ \leftarrow \mathbf{v3} \end{matrix}$$

Length of $\mathbf{v1}$ = sqrt($\mathbf{v1}\cdot\mathbf{v1}$)

Unitary Matrix

Rotation matrix is a **unitary matrix.**

Any unitary matrix can be used to represent a rotation

- Might be around an arbitrary axis though.
- Will show you the intuition later.

$$Q^T = Q^{-1} \qquad Q\,Q^T = I$$

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \leftarrow \mathbf{v1} \\ \leftarrow \mathbf{v2} \\ \leftarrow \mathbf{v3} \end{matrix}$$

Length of $\mathbf{v1}$ = sqrt($\mathbf{v1}\cdot\mathbf{v1}$)
= $\cos^2\theta + \sin^2\theta + 0 = 1$

(De)Composing Transformations

Combining Transformations

For instance: rotate P around the z-axis, then around y-axis, and scale it.

Generally, combining transformations can be done by multiplying individual transformation matrices together first to derive a composite matrix, which is then applied once in the end.

1. **First rotation: $P_{t1} = P \times T_z$**
2. **Second rotation: $P_{t2} = P_{t1} \times T_y$**
3. **Scaling: $P' = P_{t2} \times T_s$**
4. **Overall: $P' = P \times T_z \times T_y \times T_s$**
5. **$P' = P \times T$**

Since matrix multiplication is **associative**, let **$T = T_z \times T_y \times T_s$**, which represents the combination effect of the three transformations

Combining Transformations

For instance: rotate P around the z-axis, then around y-axis, and scale it.

Generally, combining transformations can be done by multiplying individual transformation matrices together first to derive a composite matrix, which is then applied once in the end.

A sequence of arbitrary rotations is still a rotation, because the product of a set of unitary matrices is still a unitary matrix. Can you prove it?

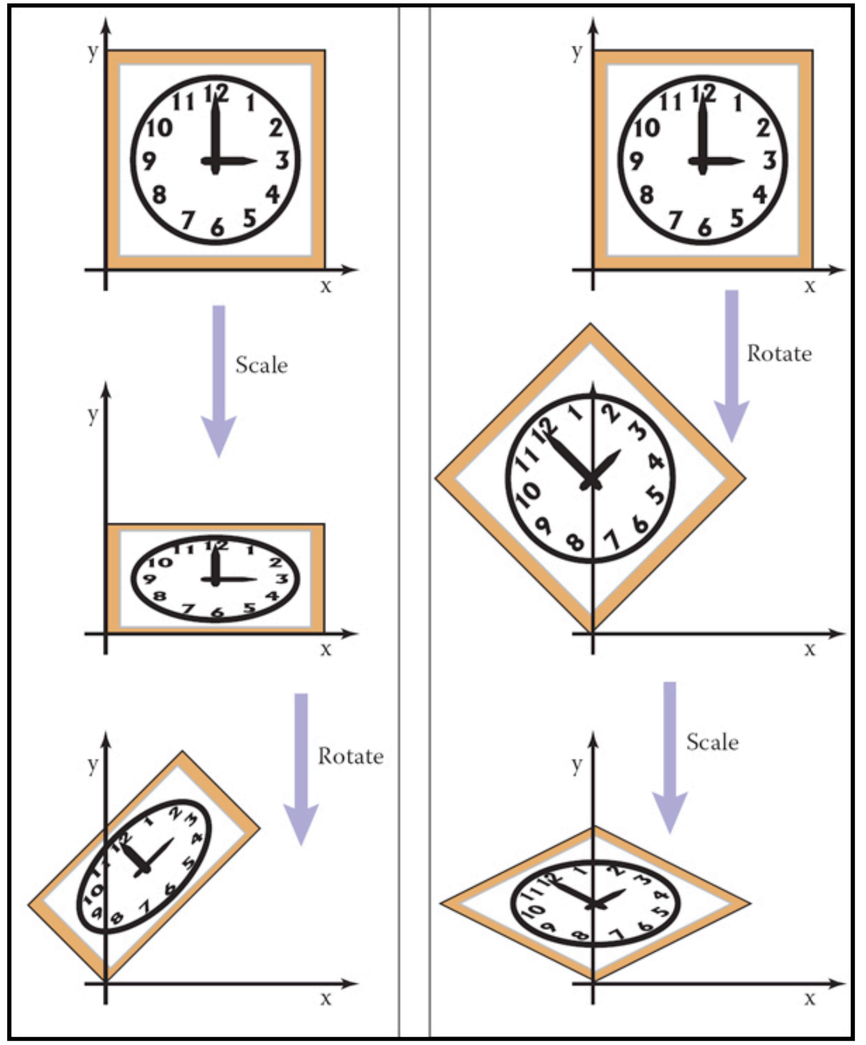
Combining Transformations

Can we reorder the individual transformations?

Is rotating P around the z-axis, then around y-axis, and scaling P the same as rotating around y, then z, then scaling P?

- $T_z \times T_y \times T_s = T_y \times T_z \times T_s$?

No. Matrix multiplication is not commutative.



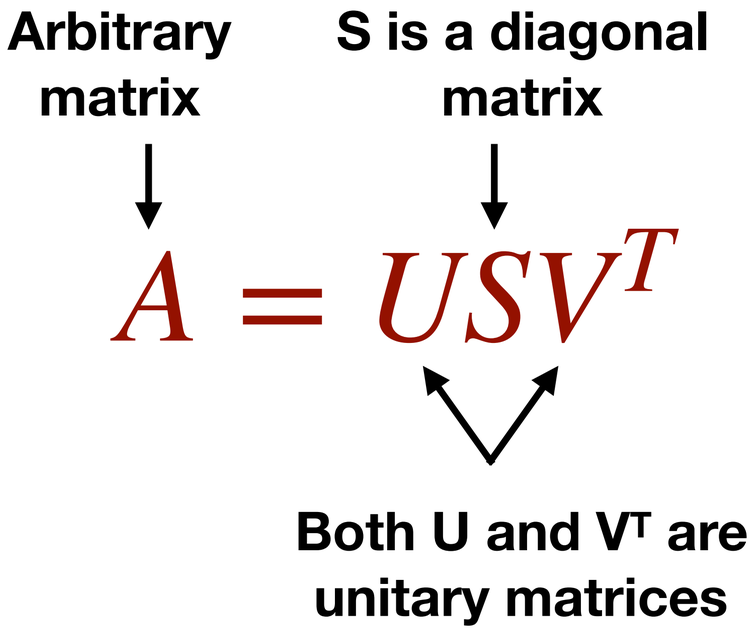
Decomposing Transformations

Can we decompose any arbitrary transformation into a sequence of basic transformations?

Yes. There are multiple ways. One common way is through **singular value decomposition** (SVD).

Any arbitrary transformation can be composed as a rotation, a scaling, and another rotation.

There are other ways to decompose a matrix and thus other ways to decompose a transformation.



Translation

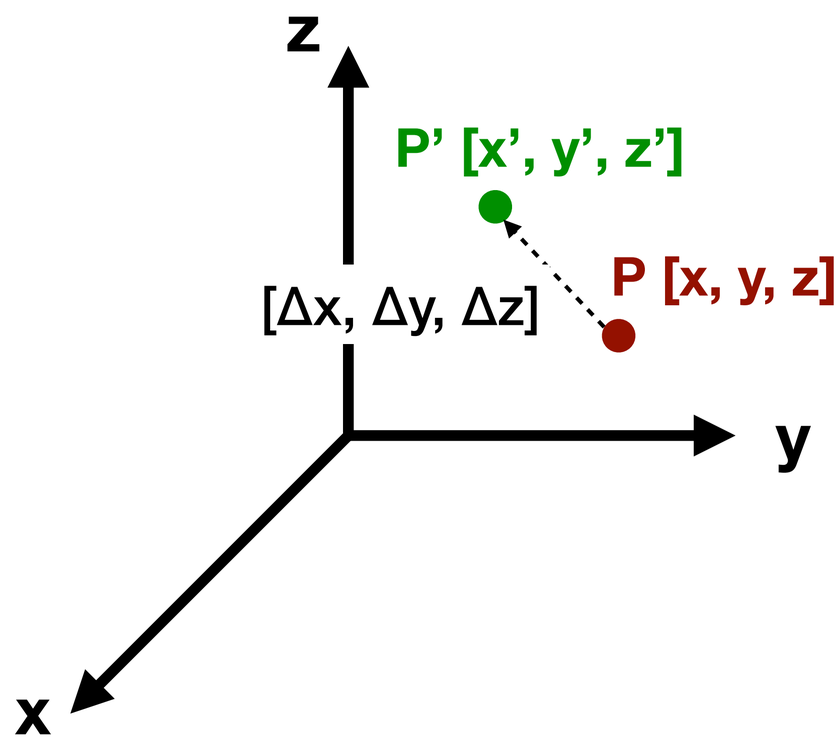
Translation

Move P [x, y, z] along the x-axis by Δx

Move P [x, y, z] along the y-axis by Δy

Move P [x, y, z] along the z-axis by Δz

P [x, y, z] becomes P’ [x + Δx, y + Δy, z + Δz]



Translation

What should the transformation matrix be if we want to move P [x, y, z] to P’ [x + Δx, y + Δy, z + Δz] regardless of where P is?

Can we treat it as scaling? What would the scaling factor be?

- $S_0 = (x + \Delta x) / x = 1 + \Delta x / x$
- $S_1 = (y + \Delta y) / y = 1 + \Delta y / y$
- $S_2 = (z + \Delta z) / z = 1 + \Delta z / z$
- The scaling factor depends on [x, y, z].
- So there is no single scaling factor that applies to all points P.
- Reducing translation to scaling isn’t a general approach.

$$\begin{bmatrix} \mathbf{1+\Delta x/x} & 0 & 0 \\ 0 & \mathbf{1+\Delta y/y} & 0 \\ 0 & 0 & \mathbf{1+\Delta z/z} \end{bmatrix}$$

Translation

What should the transformation matrix be?

$$x' = xT_{00} + yT_{10} + zT_{20} = x + \Delta x$$

A 3x3 matrix can't express the Δx term!

$$[x, y, z] \quad x \quad \begin{bmatrix} T_{00}, T_{01}, T_{02} \\ T_{10}, T_{11}, T_{12} \\ T_{20}, T_{21}, T_{22} \end{bmatrix} \quad = \quad [x', y', z']$$

Translation

We could make it work by adding one new term: T_{30}

$$x' = xT_{00} + yT_{10} + zT_{20} + T_{30} = x + \Delta x$$

$$1 \qquad 0 \qquad 0 \qquad \Delta x$$

$$[x, y, z] \quad x \quad \begin{bmatrix} T_{00}, T_{01}, T_{02} \\ T_{10}, T_{11}, T_{12} \\ T_{20}, T_{21}, T_{22} \end{bmatrix} \quad = \quad [x', y', z']$$

Translation

Effectively, the matrix becomes 4x3, and P needs to be 1x4.

$$x' = xT_{00} + yT_{10} + zT_{20} + T_{30} = x + \Delta x$$

$$1 \qquad 0 \qquad 0 \qquad \Delta x$$

$$[x, y, z, 1] \quad x \quad \begin{bmatrix} 1 & T_{01}, T_{02} \\ 0 & T_{11}, T_{12} \\ 0 & T_{21}, T_{22} \\ \Delta x & T_{31}, T_{32} \end{bmatrix} = [x', y', z']$$

Translation

But, P’ is still 1x3, which prevents **further translations** on P’!

So P’ needs to be 1x4 as well, which means T needs to be 4x4.

$$[x, y, z, 1] \quad x \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \Delta x & \Delta y & \Delta z \end{bmatrix} \quad = \quad [x', y', z']$$

Translation

What should the additional column be?

$$xT_{03} + yT_{13} + zT_{23} + T_{33} = 1, \text{ for } \forall\ x, y, z$$

$$0 \quad 0 \quad 0 \quad 1$$

$$[x, y, z, 1] \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \Delta x & \Delta y & \Delta z & 1 \end{bmatrix} = [x', y', z', 1]$$

Homogeneous Coordinates

Homogeneous Coordinates

[x, y, z] is the **cartesian coordinates** of P.

[x, y, z, 1] is the **homogeneous coordinates** of P.

Homogeneous coordinates are introduced so that translation could be expressed as matrix multiplication.

$$[x, y, z, 1] \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \Delta x & \Delta y & \Delta z & 1 \end{bmatrix} = [x', y', z', 1]$$

Homogeneous Coordinates

For translation to work:

- The last element in the homogeneous coordinates has to be 1.
- The last column of the matrix has to be $[0, 0, 0, 1]^T$ (We will see what would happen if this is not the case later in the semester when we talk about perspective transformations.)

But do they generally apply to other transformations?

$$[x, y, z, 1] \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \Delta x & \Delta y & \Delta z & 1 \end{bmatrix} = [x', y', z', 1]$$

The Identity Matrix in Homogeneous Coordinates

The top-left 3x3 sub-matrix is the same identity matrix as before.

$$[x, y, z, 1] \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [x', y', z', 1]$$

Scaling in Homogeneous Coordinates

Scaling P [x, y, z, 1] to P' [S_0.x, S_1.y, S_2.z, 1].

The top-left 3x3 sub-matrix is the same as before.

$$[x, y, z, 1] \times \begin{bmatrix} S_0 & 0 & 0 & 0 \\ 0 & S_1 & 0 & 0 \\ 0 & 0 & S_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [x', y', z', 1]$$

Rotation in Homogeneous Coordinates

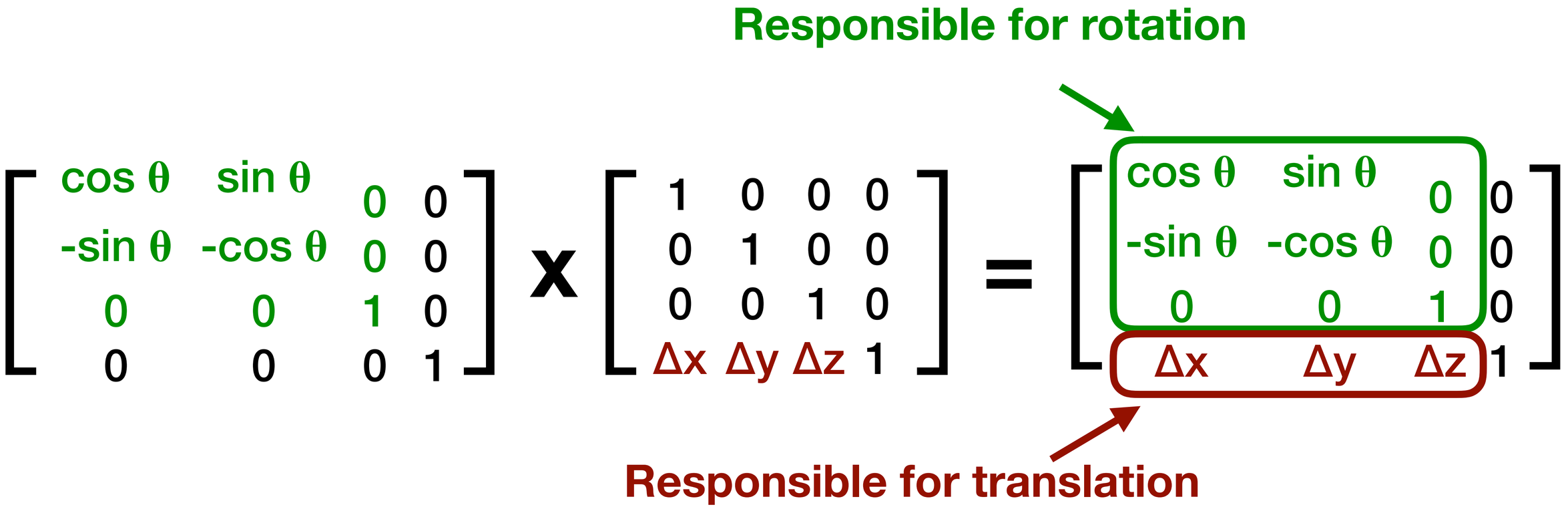
Rotate P around the z-axis by θ?

The top-left 3x3 sub-matrix is the same as before.

$$[x, y, z, 1] \times \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [x', y', z', 1]$$

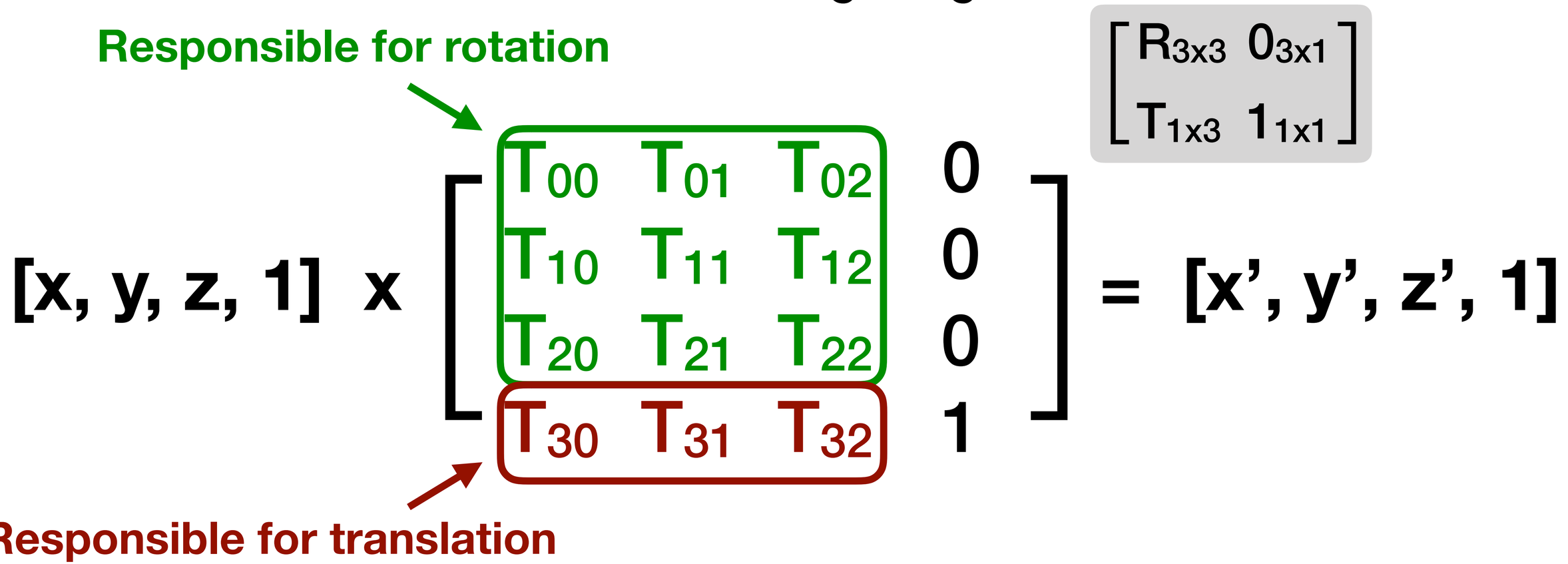
Composite Transformation in Homogeneous Coordinates

Rotate P about the z-axis by θ and translate by $[\Delta x, \Delta y, \Delta z]$.



Composite Transformation in Homogeneous Coordinates

Transforming with the composite matrix is equivalent to first rotating using the rotation sub-matrix and then translating using the translation sub-matrix.



Affine Transformation

Matrix that has this form (last column vector is [0, 0, 0, 1]) is called an affine transformation matrix.

Intuitively, affine transformation preserves straight lines and line parallelism.

- Translation, scale, rotation all do not bend straight lines and preserve parallelism.
- Are camera projections affine?

$$\begin{bmatrix} T_{00} & T_{01} & T_{02} & 0 \\ T_{10} & T_{11} & T_{12} & 0 \\ T_{20} & T_{21} & T_{22} & 0 \\ T_{30} & T_{31} & T_{32} & 1 \end{bmatrix}$$

https://www.adorama.com/alc/10-must-have-gadgets-for-architecture-photography/

https://www.lifewire.com/what-is-a-fisheye-lens-4774336

Cartesian-Homogeneous Coordinates Conversion

[x, y, z] <==> [x, y, z, 1]

In fact, [x, y, z] <==> [kx, ky, kz, k]

If [x, y, z, 1] after transformation T becomes [x', y', z', 1], then [kx, ky, kz, k] after the same transformation T will become [kx', ky', kz', k], because T is linear (matrix multiplication).

- In this case, we get the Cartesian coordinates by [kx'/k, ky'/k, kz'/k, k/k].
- Usually k is 1, but k could be set to other values later (e.g., in perspective transformation).
- The kx, ky, kz, and k in [kx, ky, kz, k] don't have physical meanings. When you convert it back to [x, y, z], it then corresponds to a point in the physical world.

Vector Transformation

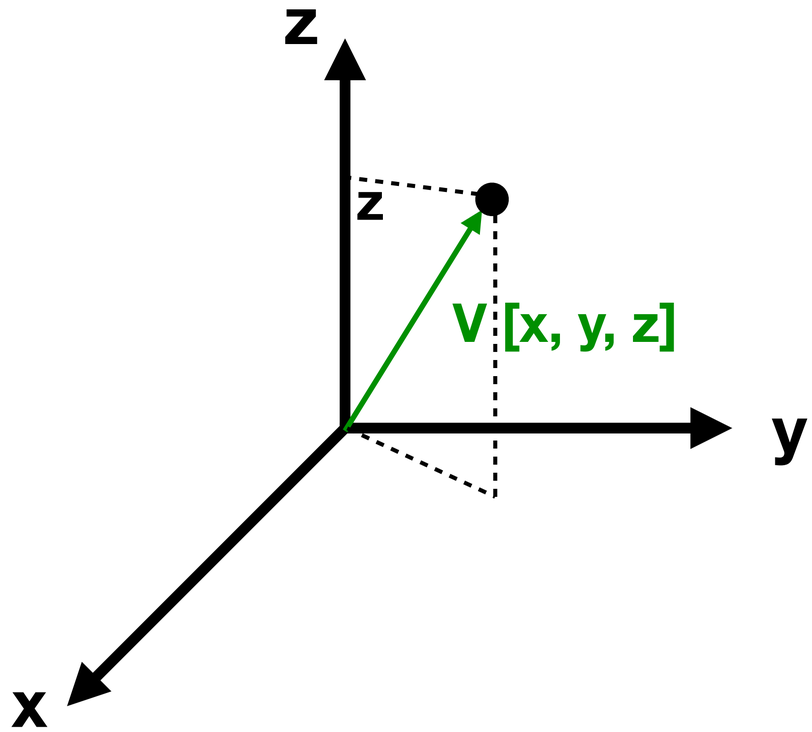
Vector

A vector has the same representation of a point: [x, y, z].

A vector represents a direction between [0, 0, 0] and [x, y, z] with a length.

- Another interpretation: vector [x, y, z] represents all points whose **coordinate ratio** is x:y:z.

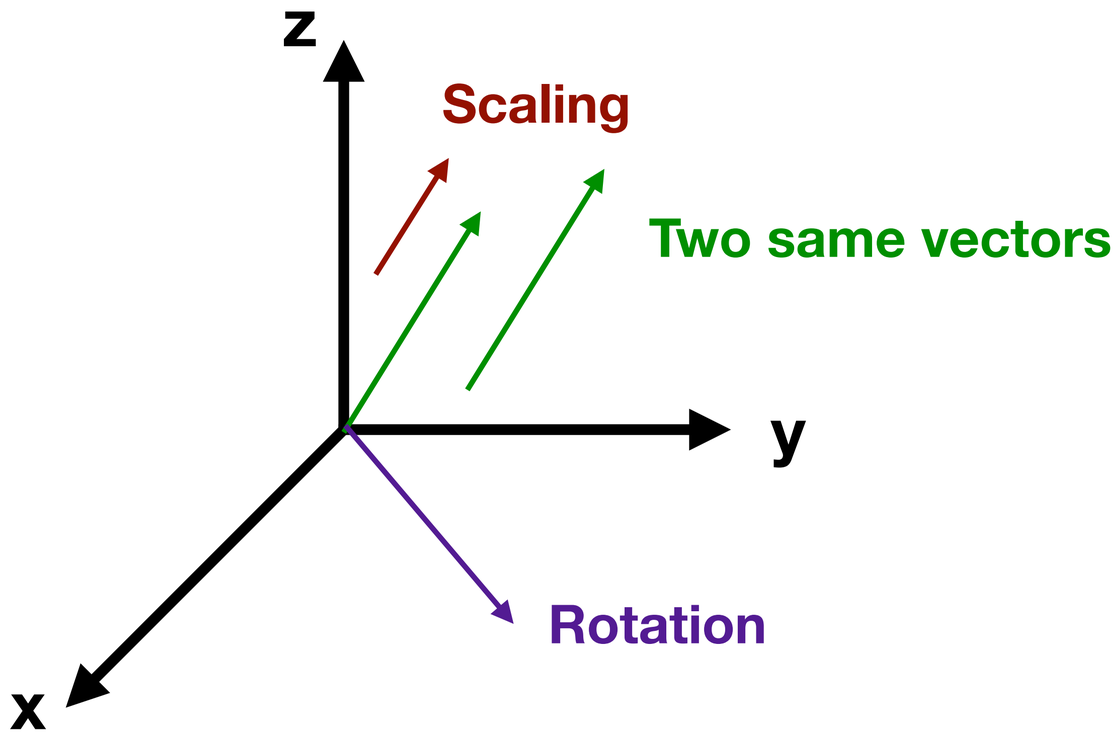
A unit vector or normalized vector is one whose length/norm **sqrt($x^2+y^2+z^2$)** is 1.



Vector

A vector is “positionless”, so translating vectors is meaningless.

Rotation and scaling are meaningful vector transformations.



Vector Transformation in Homogeneous Coordinates

Vector and point transformations are almost the same, but:

V [x, y, z] in Cartesian coordinates is [x, y, z, **0**] in homogeneous coordinates.
0 ensures that translation doesn't change the vector.

The homogeneous transformation matrix is the same.

$$[x, y, z, 0] \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \Delta x & \Delta y & \Delta z & 1 \end{bmatrix} = [x', y', z', 0]$$

Vector Transformation in Homogeneous Coordinates

Rotate V [x, y, z] around z-axis by θ.

Same transformation matrix as before. The only difference is that the last element in the homogeneous coordinate is 0 now.

$$[x, y, z, 0] \times \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [x', y', z', 0]$$

Coordinate System Transformation

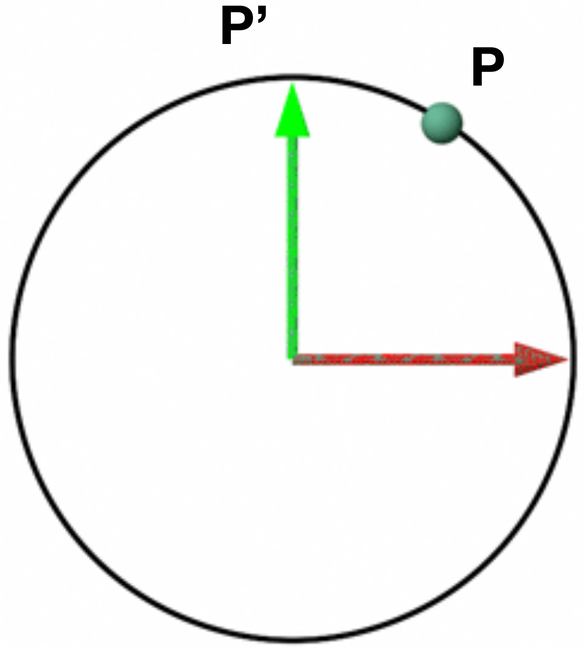
Another Way to Think About Point Transformation

Two equivalent ways to interpret rotating P to P’.

First: rotating P in the current coordinate system (a.k.a., frame) F_0 by a matrix **R**.

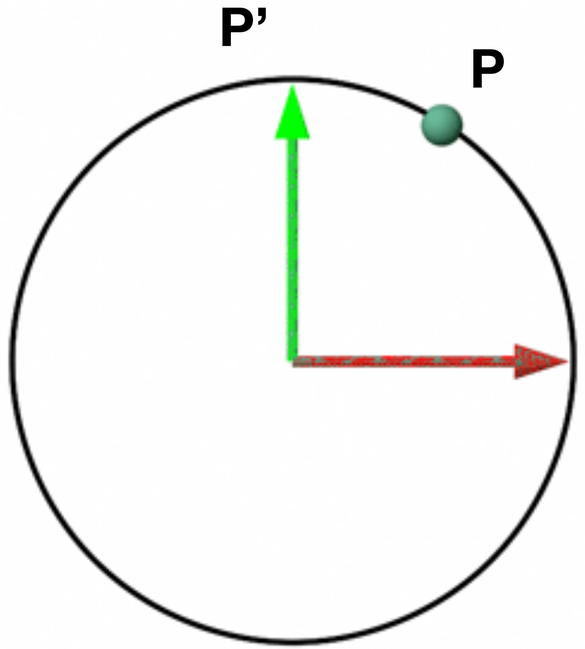
Second, rotating the current frame F_0 to a new frame F_1 using **R** while keeping the relative position of P unchanged.

- That is, the coordinates of P in F_1 are the same as those in F_0.



Another Way to Think About Point Transformation

$$[P_x, P_y, P_z, 0] \times T = [P_x', P_y', P_z', 0]$$



Two ways to interpret T

- Transform $[P_x, P_y, P_z, 0]$ to $[P_x', P_y', P_z', 0]$ in the current frame F_0 using matrix T.
- Transform the current frame F_0 to a new frame F_1 using matrix T and keep the coordinates $[P_x, P_y, P_z, 0]$.

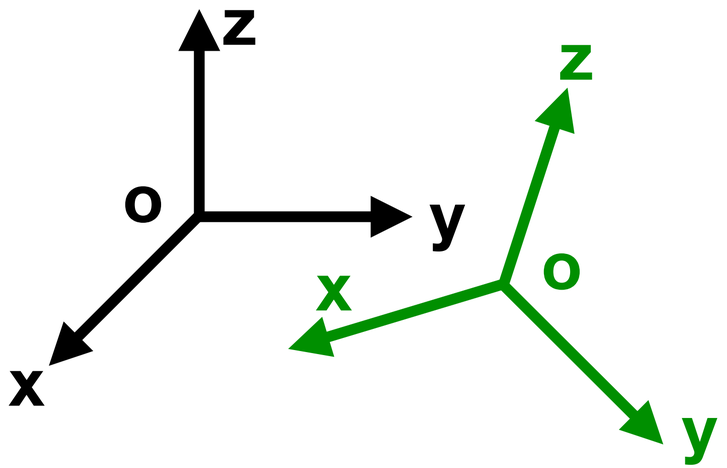
What does it mean to transform a coordinate system?

How to Transform a Frame/Coordinate System?

A Cartesian coordinate system/frame is define by its **origin** [0, 0, 0] and three **basis vectors**: the x axis [1, 0, 0], y axis [0, 1, 0] and z axis [0, 0, 1].

When we create a new frame, we can think of it transforming three original basis vectors and the origin to three new basis vectors and a new origin.

4 transformations (3 vector transformations + 1 point transformation).



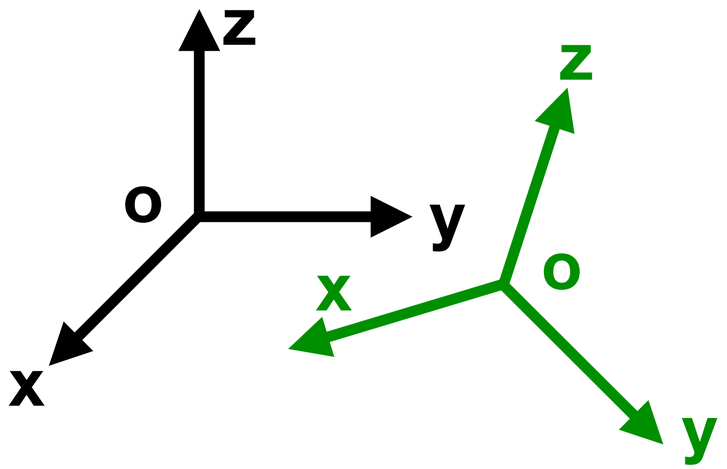
How to Transform a Frame/Coordinate System?

One single transformation matrix can express all 4 transformations. How?

$$[1, 0, 0, 0] \times \begin{bmatrix} T_{00}, T_{01}, T_{02}, 0 \\ T_{10}, T_{11}, T_{12}, 0 \\ T_{20}, T_{21}, T_{22}, 0 \\ T_{30}, T_{31}, T_{32}, 1 \end{bmatrix} = [T_{00}, T_{01}, T_{02}, 0]$$

Original x-axis

This is the new x-axis.
Only the first row is used
to transform x-axis.

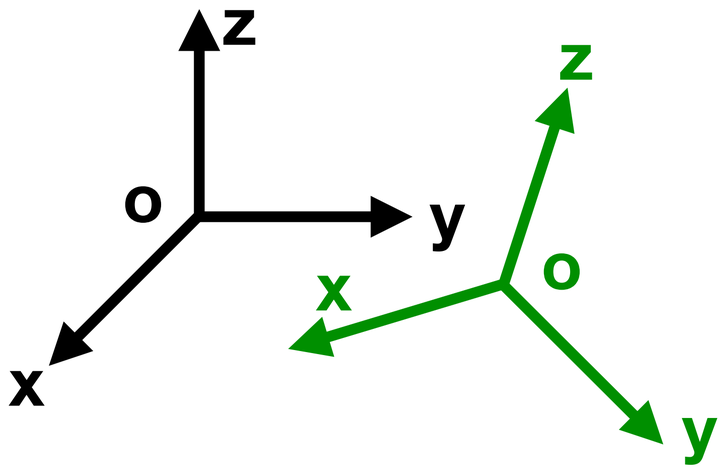


How to Transform a Frame/Coordinate System?

$$[0, 1, 0, 0] \times \begin{bmatrix} T_{00}, T_{01}, T_{02}, 0 \\ T_{10}, T_{11}, T_{12}, 0 \\ T_{20}, T_{21}, T_{22}, 1 \\ T_{30}, T_{31}, T_{32}, 1 \end{bmatrix} = [T_{10}, T_{11}, T_{12}, 0]$$

Original y-axis

This is the new y-axis.
Only the second row is used
to transform y-axis.

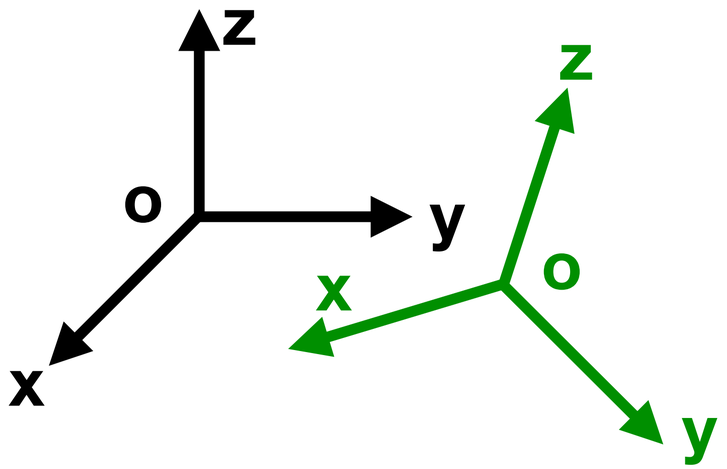


How to Transform a Frame/Coordinate System?

$$[0, 0, 1, 0] \times \begin{bmatrix} T_{00}, T_{01}, T_{02}, 0 \\ T_{10}, T_{11}, T_{12}, 0 \\ T_{20}, T_{21}, T_{22}, 0 \\ T_{30}, T_{31}, T_{32}, 1 \end{bmatrix} = [T_{20}, T_{21}, T_{22}, 0]$$

Original z-axis

This is the new z-axis.
Only the third row is used
to transform z-axis.

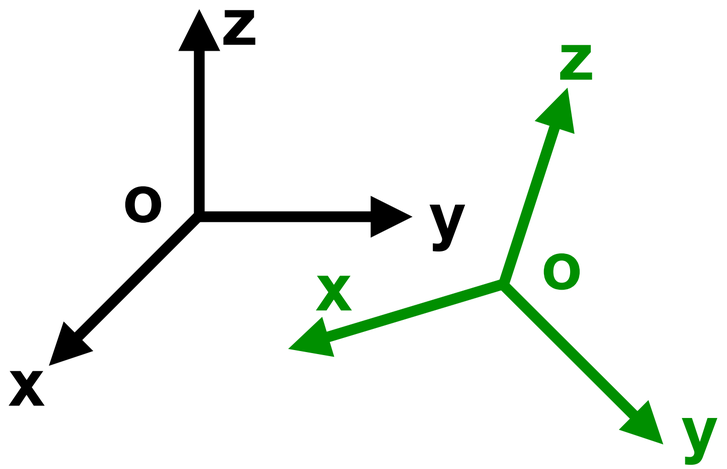


How to Transform a Frame/Coordinate System?

$$[0, 0, 0, 1] \times \begin{bmatrix} T_{00}, T_{01}, T_{02}, 0 \\ T_{10}, T_{11}, T_{12}, 0 \\ T_{20}, T_{21}, T_{22}, 0 \\ T_{30}, T_{31}, T_{32}, 1 \end{bmatrix} = [T_{30}, T_{31}, T_{32}, 1]$$

Original origin

This is the new origin.
Only the fourth row is used
to transform origin.



How to Transform a Frame/Coordinate System?

The transformation matrix directly encodes the new basis vectors and the new origin!

The last column needs to be $[0, 0, 0, 1]^T$

The identity matrix basically encodes the original frame.

$$\begin{bmatrix} T_{00}, T_{01}, T_{02}, 0 \\ T_{10}, T_{11}, T_{12}, 0 \\ T_{20}, T_{21}, T_{22}, 0 \\ T_{30}, T_{31}, T_{32}, 1 \end{bmatrix}$$

← New x-axis basis vector

← New y-axis basis vector

← New z-axis basis vector

← New origin

All in homogeneous coordinates

Identity matrix encodes the canonical frame's information!

$$\begin{bmatrix} 1, 0, 0, 0 \\ 0, 1, 0, 0 \\ 0, 0, 1, 0 \\ 0, 0, 0, 1 \end{bmatrix}$$

Back to Rotation Matrix Being Unitary Matrix

This provides an intuitive explanation why a rotation matrix must be unitary.

- The top-left 3x3 matrix simultaneously serves two roles:1) it encodes the new basis vectors in the new coordinate system, and 2) it encodes the rotation matrix. For the first role, it must be a unitary matrix; so the rotation matrix must be a unitary matrix.

$$\begin{bmatrix} T_{00}, T_{01}, T_{02}, 0 \\ T_{10}, T_{11}, T_{12}, 0 \\ T_{20}, T_{21}, T_{22}, 0 \\ T_{30}, T_{31}, T_{32}, 1 \end{bmatrix}$$

← New x-axis basis vector

← New y-axis basis vector

← New z-axis basis vector

← New origin

All in homogeneous coordinates

Identity matrix encodes the original frame's basic vectors and the origin!

$$\begin{bmatrix} 1, 0, 0, 0 \\ 0, 1, 0, 0 \\ 0, 0, 1, 0 \\ 0, 0, 0, 1 \end{bmatrix}$$

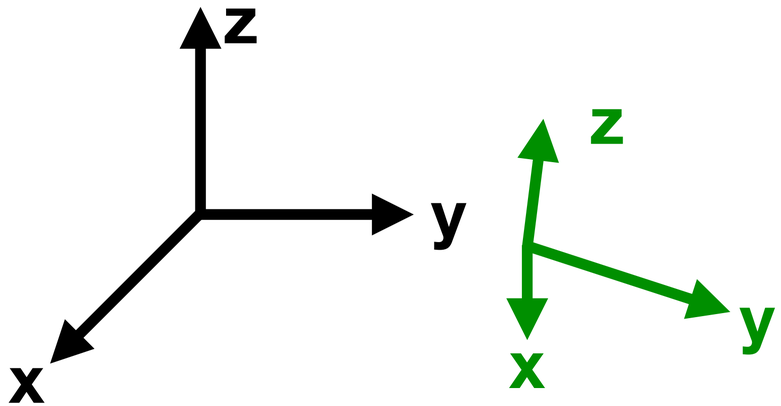
How to Transform a Frame/Coordinate System?

Does any transformation matrix work?

- Yes, but for the transformed frame to be used as a Cartesian coordinate system, the top 3x3 matrix must be an orthogonal matrix: the three basic vectors must be mutually orthogonal and their lengths must be 1.

Intuition: an orthogonal matrix rotates the three basis vector together, so mutual orthogonality and unit length requirements are naturally met.

$$\begin{bmatrix} 2, 4, 0, 0 \\ 1, 5, -1, 0 \\ -1, 0, 1, 0 \\ 1, 4, 0, 1 \end{bmatrix}$$



A legal transformation of the frame, but the new frame can't be used as a Cartesian coordinate system.

Rotation Around an Arbitrary Axis (w here)

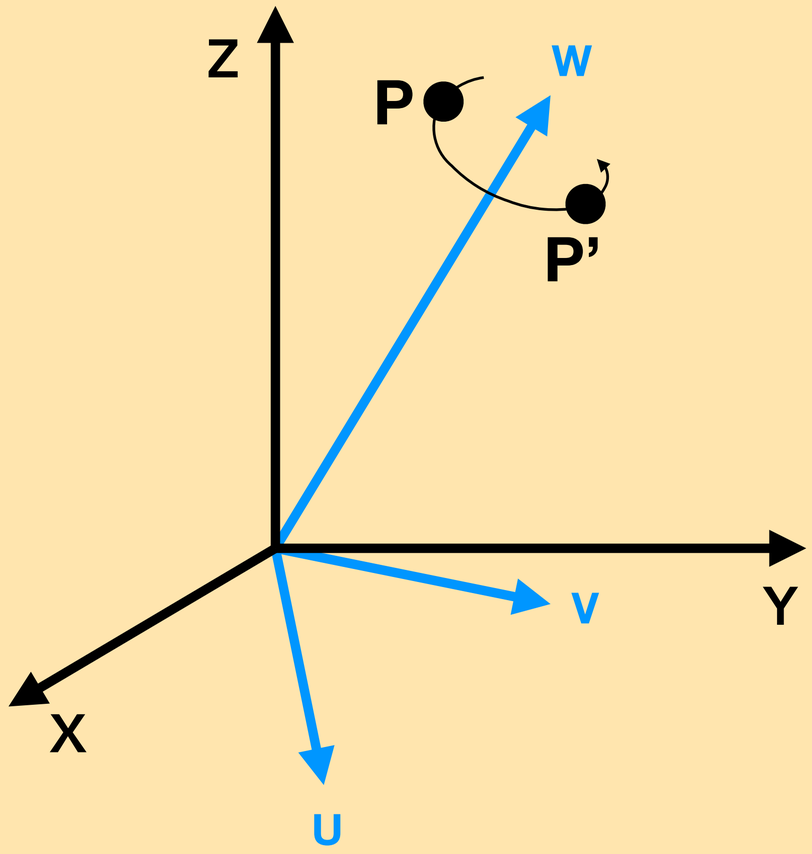
First, create a Cartesian coordinate system **UVW**. There are infinite many (since only **W** is given); any one will work in principle.

Second, rotate **UVW** to be **XYZ**; let the rotation matrix be R_1. P becomes P1.

Third, rotate P1 around **Z**. Let the rotation matrix be R_2. P1 becomes P2.

Finally, rotate P2 from **XYZ** to **UVW** to get P'. The rotation matrix is necessarily R_1^{-1} which is R_1^T since R_1 is necessarily orthogonal.

$$P' = P \times R_1 \times R_2 \times R_1^{-1}$$



https://www.researchgate.net/figure/Notation-used-for-the-analysis-of-the-rotation-around-an-arbitrary-axis-r_fig5_230934580