CSC 576 - 2015 Fall: Homework 1

Hand in hardcopy before the class on Sep. 22

Requirements

LaTeX generated Homework is preferred. One additional point can be obtained if your homework is created from LaTeX. Due to the request from some students, the homework is posted online right now, but would be updated probably every week until it is formally released. (*), (**), or (***) indicates the difficulty of each question. Please hand in your homework in the form of hardcopy before our class on Sep. 22.

1 (*) Question 1: 2 points

Verify that the function \( f(x) = \max_i |x_i| \) is a norm.

2 (*) Question 2: 2 points

Prove the following inequality

\[
\forall x, y \in \mathbb{R}^n \quad |\langle x, y \rangle| \leq \|x\|_1 \|y\|_\infty.
\]

(You are not allowed to use Holder Inequality). Please prove it from the basic definition.

3 (*) Question 3: 2 points

Prove the following inequality

\[
\forall x \in \mathbb{R}^n \quad \|x\|_2 \leq \sqrt{n} \|x\|_\infty.
\]

4 (*) Question 4: 2 points

Consider the linear equation \( Ax = b \). Give an example for “\( A \)” and “\( b \)” to satisfy the following four situations respectively:

- these exists one and only one solution;
- these does not exist any solution;
- these exists more than one solution;
- \( A \) is not invertible and these exists one and only one solution.
5 (*) Question 5: 2 points
Prove the following two statements:
• \( \{ L(x) \mid x \in S \} \) is a linear space if \( S \) is a linear space and \( L \) is a linear transformation.
• \( \{ x \mid L(x) \in S \} \) a linear space assuming \( S \) is a linear space, and \( L \) is a linear transformation.

6 (*) Question 6: 3 points
Compute the gradient (derivative) in terms of \( X \in \mathbb{R}^{m \times n} \) for the following functions:
• \( f(X) = \text{trace}(AXB) \);
• \( f(X) = \text{trace}(AXBXC) \);
• \( f(X) = \frac{1}{2} \| AXB - C \|_F^2 \).

7 (**) Question 7: 2 points
Given three square matrices \( A, B, C \in \mathbb{R}^{n \times n} \). Apply Randomized Kaczmarz algorithm to find a matrix \( X \in \mathbb{R}^{n \times n} \) satisfying the following linear equation:
\[
AX + XB = C. 
\] 
Assume that these exists a least solution. Write down the algorithm procedure, that is, how to update \( X_{k+1} \) from \( X_k \). (Hint: you can think about first how many linear equations defined in [1].)

8 (***) Question 8: 3 points
Consider a linear system \( Ax = b \). Assume that there exists at least one solution. The randomized Kaczmarz (RK) algorithm initializes \( x_0 \) as 0, resulting in a convergent point \( A^+b \) for the sequence \( \{x_k\} \). Now assuming that start from an arbitrary initial point \( x_0 \) and run the RK algorithm. My questions are
• (1 point) Does it converge?
• (2 points) If not, please give an example. If yes, please write down the closed form of the convergent point starting from an arbitrary initial point \( x_0 \). (Hint: starting from an arbitrary point \( x_0 \) is equivalent to moving all hyperplanes along \( -x_0 \) and starting from 0.)

9 (***) Question 9: 2 points
Define the matrix \( \ell_{p,q} \) norm for \( A \in \mathbb{R}^{m \times n} \) as
\[
\| A \|_{p,q} := \left( \sum_i \left( \sum_j |A_{ij}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}},
\] 
where \( p, q \geq 1 \). Please provide its dual norm and prove it.